THE STANDARD MODEL

Adrian Signer

Institute for Particle Physics Phenomenology
University of Durham
Durham DH1 3LE

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Introduction

The Standard Model is a description of the strong, weak and electromagnetic interactions in terms of "gauge theories". A gauge theory is one that possesses invariance under a set of "local transformations" i.e. transformations whose parameters are space-time dependent.

Electromagnetism is a well-known example of a gauge theory. In this case the gauge transformations are local complex phase transformations of the fields of charged particles, and gauge invariance necessitates the introduction of a massless vector (spin-1) particle, called the photon, whose exchange mediates the electromagnetic interactions.

In the 1950's Yang and Mills considered (as a purely mathematical exercise) extending gauge invariance to include local non-abelian (i.e. non-commuting) transformations such as $SU(2)$. In this case one needs a set of massless vector fields (three in the case of $SU(2)$), which were formally called "Yang-Mills" fields, but are now known as "gauge fields".

In order to apply such a gauge theory to weak interactions, one considers particles which transform into each other under the weak interactions, such as a u-quark and a d-quark, or an electron and a neutrino to be arranged in doublets of weak isospin. The three gauge bosons are interpreted as the $W^{\pm}$ and Z bosons, that mediate weak interactions in the same way that the photon mediates electromagnetic interactions.

The difficulty in the case of weak interactions was that the weak interactions are known to be short range, mediated by very massive vector bosons, whereas Yang-Mills fields are required to be massless in order to preserve the gauge invariance. The apparent paradox was solved by the application of the "Higgs mechanism". This is a prescription for breaking the gauge symmetry spontaneously. In this scenario one starts with a theory that possesses the required gauge invariance, but the ground state of the theory is *not* invariant under the gauge transformations. The breaking of the invariance arises in the quantization of the theory, whereas the Lagrangian only contains terms which *are* invariant. One of the consequences of this is that the gauge bosons acquire a mass and the theory can thus be applied to weak interactions.

Spontaneous symmetry breaking and the Higgs mechanism has another extremely important consequence. It leads to a renormalizable theory with massive vector bosons. This means that one can carry out a programme of renormalization in which the infinities that arise in higher-order calculations can be reabsorbed into the parameters of the Lagrangian (as in the case of QED). Had one simply broken the gauge invariance explicitly by adding mass terms for the gauge bosons the resulting theory would not have been renormalizable and could not therefore have been used to carry out perturbative calculations. A consequence of the Higgs mechanism is the existence of a scalar (spin-0) particle, the Higgs boson.

The remaining step was to apply the ideas of gauge theories to the strong interactions. The gauge theory of strong interactions is called "Quantum ChromoDynamics" (QCD). In this theory the quarks possess an internal property called "color" and the gauge transformations are local transformations between quarks of different colors. The gauge bosons of QCD are called "gluons" and they mediate the strong interactions.

The union of QCD and the electroweak gauge theory, which describes the weak and electromagnetic interactions is known as the Standard Model. It has a very simple structure and the different forces of nature are treated in the same fashion, i.e. as gauge theories. It has eighteen fundamental parameters, most of which are associated with the masses of the gauge bosons, the quarks and leptons, and the Higgs. Nevertheless these are not all independent and, for example, the ratio of the W and Z boson masses are (correctly) predicted by the model. Since the theory is renormalizable, perturbative calculations can be performed that predict cross sections and decay rates both for strongly and weakly interacting processes. These predictions have met with considerable success when confronted with experimental data, which is becoming more and more precise, therefore posing increasingly stringent tests on the Standard Model.

1 QED as an Abelian Gauge Theory

1.1 Gauge Transformations

Consider the Lagrangian density for a free Dirac field ψ:

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi \tag{1.1}$$

This Lagrangian density is invariant under a phase transformation of the fermion field

$$\psi \;\rightarrow\; e^{-i\omega}\psi, \qquad \overline{\psi} \;\rightarrow\; e^{i\omega}\overline{\psi}, \tag{1.2}$$

where ω is a real constant (i.e. independent of x) and $\overline{\psi}$ is the conjugate field.

The set of all numbers $e^{-i\omega}$ form a group[1]. This particular group is "abelian" which is to say that any two elements of the group commute. This just means that

$$e^{-i\omega_1}e^{-i\omega_2} \;=\; e^{-i\omega_2}e^{-i\omega_1}. \tag{1.3}$$

This particular group is called $U(1)$ which means the group of all unitary 1×1 matrices. A unitary matrix satisfies $U^+ = U^{-1}$ with U^+ being the adjoint matrix.

We can now state the invariance of the Lagrangian eq. (1.1) under phase transformations in a more fancy way by saying that the Lagrangian is invariant under global $U(1)$ transformations. By global we mean that ω does not depend on x.

For the purposes of these lectures it will usually be sufficient to consider infinitesimal group transformations, i.e. we assume that the parameter ω is sufficiently small that we can expand in ω and neglect all but the linear term. Thus we write

$$e^{-i\omega} = 1 \;-\; i\,\omega \;+\; \mathcal{O}(\omega^2). \tag{1.4}$$

Under such infinitesimal phase transformations the field ψ changes according to

$$\psi \rightarrow \psi + \delta\psi = \psi - \; i\,\omega\,\psi, \tag{1.5}$$

and the conjugate field $\overline{\psi}$ by

$$\overline{\psi} \rightarrow \overline{\psi} + \delta\overline{\psi} \;\; = \overline{\psi} + \, i\,\omega\,\overline{\psi}, \tag{1.6}$$

such that the Lagrangian density remains unchanged (to order ω).

[1] A group is a mathematical term for a set, where multiplication of elements is defined and results in another element of the set. Furthermore, there has to be a 1 element (s.t. $1 \times a = a$) and an inverse (s.t. $a \times a^{-1} = 1$) for each element a of the set

At this point we should note that global transformations are not very attractive from a theoretical point of view. The reason is that making the same transformation at every space-time point requires that all these points 'know' about the transformation. But if I make a certain transformation at the top of the Matterhorn, how can a point somewhere in England know about it? It would take some time for a signal to travel from the Matterhorn to England (and frankly, which signal would be crazy enough to leave the Matterhorn for England).

Thus, we have two options at this point. Either, we simply note the invariance of eq. (1.1) under global $U(1)$ transformations and put this aside as a curiosity, or we insist that invariance under gauge transformations is a fundamental property of nature. If we take the latter option we have to require invariance under local transformations. Local means that the parameter of the transformation, ω, now depends on the space-time point x. Such local (i.e. space-time dependent) transformations are called "gauge transformations".

If the parameter ω depends on the space-time point then the field ψ transforms as follows under infinitesimal transformations

$$\delta\psi(x) = -i\,\omega(x)\,\psi(x); \qquad \delta\overline{\psi}(x) = i\,\omega(x)\,\overline{\psi}(x). \tag{1.7}$$

Note that the Lagrangian density eq. (1.1) now is *no longer* invariant under these transformations, because of the partial derivative that is interposed between $\overline{\psi}$ and ψ. This derivative will act on the space-time dependent parameter $\omega(x)$ such that the Lagrangian density changes by an amount $\delta\mathcal{L}$, where

$$\delta\mathcal{L} \ = \overline{\psi}(x)\,\gamma^\mu\,[\partial_\mu\omega(x)]\,\psi(x). \tag{1.8}$$

The square brackets in $[\partial_\mu\,\omega(x)]$ are introduced to indicate that the derivative ∂_μ acts only inside the brackets. It turns out that we can restore gauge invariance if we assume that the fermion field interacts with a vector field A_μ, called a "gauge field", with an interaction term

$$-\,e\,\overline{\psi}\,\gamma^\mu A_\mu \psi \tag{1.9}$$

added to the Lagrangian density which now becomes

$$\mathcal{L} = \overline{\psi}\ \left(i\gamma^\mu\left(\partial_\mu + i\,e\,A_\mu\right) - m\right)\psi. \tag{1.10}$$

In order for this to work we must also assume that apart from the fermion field transforming under a gauge transformation according to eq. (1.7) the gauge field, A_μ, also changes according to

$$A_\mu \to A_\mu + \delta A_\mu(x) \ = \ A_\mu + \frac{1}{e}[\partial_\mu\,\omega(x)]. \tag{1.11}$$

Exercise 1.1
Using eqs. (1.7) and (1.11) show that under a gauge transformation $\delta(-e\,\overline{\psi}\,\gamma^\mu A_\mu \psi) \;=\; -\overline{\psi}(x)\,\gamma^\mu\,[\partial_\mu \omega(x)]\,\psi(x)$.

This change exactly cancels with eq. (1.8), so that once this interaction term has been added the gauge invariance is restored. We recognize eq. (1.10) as being the fermionic part of the Lagrangian density for QED, where e is the electric charge of the fermion and A_μ is the photon field.

In order to have a proper quantum field theory, in which we can expand the photon field, A_μ, in terms of creation and annihilation operators for photons, we need a kinetic term for the field, A_μ, i.e. a term which is quadratic in the derivative of the field. Without such a term the Euler-Lagrange equation for the gauge field would be an algebraic equation and we could use it to eliminate the gauge field altogether from the Lagrangian. We need to ensure that in introducing a kinetic term we do not spoil the invariance under gauge transformations. This is achieved by defining the field strength, $F_{\mu\nu}$, as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.12}$$

where the derivative is understood to act on the A-field only[2] It is easy to see that under the gauge transformation eq. (1.11) each of the two terms on the right hand side of eq. (1.12) changes, but the changes cancel out. Thus we may add to the Lagrangian any term which depends on $F_{\mu\nu}$ (and which is Lorentz invariant, thus, with all Lorentz indices contracted). Such a term is $aF_{\mu\nu}F^{\mu\nu}$. This gives the desired term quadratic in the derivative of the field A_μ. If we choose the constant a to be $-1/4$ then the Lagrange equations of motion match exactly the (relativistic formulation) of Maxwell's equations[3].

We have thus arrived at the Lagrangian density for QED, but from the viewpoint of demanding invariance under $U(1)$ gauge transformations rather than starting with Maxwell's equations and formulating the equivalent quantum field theory.

The Lagrangian density for QED is:

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,\left(i\gamma^\mu\left(\partial_\mu + i\,e\,A_\mu\right) - m\right)\psi. \tag{1.13}$$

[2] thus strictly speaking we should write $F_{\mu\nu} = [\partial_\mu A_\nu] - [\partial_\nu A_\mu]$

[3] The determination of this constant a is the *only* place that a match to QED has been used. The rest of the Lagrangian density is obtained purely from the requirement of local $U(1)$ invariance. A different constant would simply mean a different normalization of the photon field.

Exercise 1.2
Starting with the Lagrangian density for QED write down the Euler-Lagrange equations for the gauge field A_μ and show that this results in Maxwell's equations

Note that we are *not* allowed to add a mass term for the photon. A term such as $M^2 A_\mu A^\mu$ added to the Lagrangian density is not invariant under gauge transformations, but would give us a transformation

$$\delta\mathcal{L} \;=\; \frac{2M^2}{e} A^\mu(x)\partial_\mu\omega(x). \tag{1.14}$$

Thus the masslessness of the photon can be understood in terms of the requirement that the Lagrangian be gauge invariant.

1.2 Gauge Fixing

In general, if the part of the action that is quadratic in some field, $\phi(x)$, is given, in terms of the Fourier transform $\tilde{\phi}(p)$ of the field $\phi(x)$ by

$$S_\phi \;=\; \int d^4p\, \tilde{\phi}(-p)\mathcal{O}(p)\tilde{\phi}(p), \tag{1.15}$$

then the propagator for the field, ϕ may be written as

$$i\,\mathcal{O}^{-1}(p). \tag{1.16}$$

In the case of QED the part of the action that is quadratic in the photon field is (in momentum space)

$$S_A \;=\; \int d^4p\, \frac{1}{2}\tilde{A}^\mu(-p)\left(-g_{\mu\nu}\,p^2 + p_\mu p_\nu\right)\tilde{A}^\nu(p) \tag{1.17}$$

Unfortunately the operator $(-g_{\mu\nu}\,p^2 + p_\mu p_\nu)$ does not have an inverse. Thus it seems we are not able to write down the propagator of the photon. We solve this problem by adding to the Lagrangian density a gauge fixing term

$$-\frac{1}{2(1-\xi)}\left(\partial_\mu A^\mu\right)^2. \tag{1.18}$$

With this term included (again in momentum space), S_A becomes

$$S_A \;=\; \int d^4p\, \frac{1}{2}\tilde{A}^\mu(-p)\left(-g_{\mu\nu}\,p^2 - \frac{\xi}{1-\xi}p_\mu p_\nu\right)\tilde{A}^\nu(p), \tag{1.19}$$

and, noting the relation

$$\left(g_{\mu\nu}\,p^2 + \frac{\xi}{1-\xi}p_\mu p_\nu\right)\left(g^{\nu\rho} - \xi\frac{p^\nu p^\rho}{p^2}\right) \;=\; p^2 g_\mu{}^\rho, \tag{1.20}$$

we see that the propagator for the photon may now be written as

$$-i\left(g_{\mu\nu}-\xi\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}. \tag{1.21}$$

The special choice $\xi = 0$ is known as the Feynman gauge. In this gauge the propagator eq. (1.21) is particularly simple and we will use it most of the time.

This procedure of gauge fixing seems to be somewhat strange. First we worked hard to get a gauge invariant Lagrangian and at the very next moment we spoil gauge invariance again by introducing a gauge fixing term.

The point is that we have to fix the gauge in order to be able to perform a calculation. Once we computed a physical quantity, the dependence on the gauge cancels. In other words, it does not matter how we fix the gauge and in particular, what value for ξ we take. The choice $\xi = 0$ is simply a matter of convenience. A more careful procedure would be to leave ξ arbitrary and check that all ξ-dependence in the final result cancels. This gives us a strong check on the calculation, however, at the price of making the computation much more tedious.

The procedure of fixing the gauge in order to be able to perform a calculation, even though the final result does not depend on how we fixed the gauge, can be understood by the following analogy. Assume we wanted to calculate some scalar quantity (say the time it takes for a point mass to get from one point to another) in our ordinary 3-dimensional Euclidean space. To do so, we choose a coordinate system, perform the calculation and get our final result. Of course, the result does not depend on how we chose the coordinate system, but in order to be able to perform the calculation we had to fix it somehow. Picking a coordinate system corresponds to fixing a gauge and the independence of the result on the coordinate system chosen corresponds to the gauge invariance of physical quantities. To take this one step further we remark that by far not all quantities are independent of the coordinate system. For example, the x-coordinate of the position of the point mass at a certain time depends on our choice. Similarly, there are important quantities that are gauge dependent. An example is the gauge boson propagator given in eq. (1.21). However, all measurable quantities are gauge invariant. This is where our analogy breaks down in that in our Euclidean example there are measurable quantities that do depend on the choice of the coordinate system.

Finally we should mention that eq. (1.18) is by far not the only way to fix the gauge but it will be sufficient for these lectures to consider gauges defined through eq. (1.18). These gauges are called covariant gauges.

1.3 Covariant Derivatives

Before leaving the abelian case, it is useful to introduce the concept of a "covariant derivative". This is not essential for abelian gauge theories, but will be an invaluable tool when we extend these ideas to non-abelian gauge theories.

The covariant derivative D_μ is defined to be

$$D_\mu \equiv \partial_\mu + i\,e\,A_\mu. \tag{1.22}$$

This has the property that given the transformations of the fermion field eq. (1.7) and the gauge field eq. (1.11) the quantity $D_\mu\psi$ transforms in the same way under gauge transformations as ψ.

Exercise 1.3
Show that under an infinitesimal gauge transformation $D_\mu\psi$ transforms as $D_\mu\psi \to D_\mu\psi + \delta(D_\mu\psi)$ with $\delta(D_\mu\psi) = -i\omega(x)D_\mu\psi$.

We may thus rewrite the QED Lagrangian density as

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi. \tag{1.23}$$

Furthermore the field strength $F_{\mu\nu}$ can be expressed in terms of the commutator of two covariant derivatives, i.e.

$$\begin{aligned} F_{\mu\nu} &= -\frac{i}{e}\,[D_\mu, D_\nu] = -\frac{i}{e}[\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + i\,e\,[A_\mu, A_\nu] \\ &= [\partial_\mu A_\nu] - [\partial_\nu A_\mu] = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{1.24}$$

where in the last line we adopted the conventional notation again and left out the square brackets. Let us stress once more that in using eq. (1.24) it has to be remembered that the derivatives act only on the A-field. Note also the difference between $[\partial_\mu, A_\nu]$ and $[\partial_\mu A_\nu]$. The first term is a commutator whereas the second indicates that the derivative acts on the A-field only and nothing else to the right of it. However (and this explains the notation) we have $[\partial_\mu, A(x)]B(x) = \partial_\mu A(x)B(x) - A(x)\partial_\mu B(x) = [\partial_\mu A(x)]B(x)$.

1.4 Summary

- It is possible for the Lagrangian for a (complex) Dirac field to be invariant under local U(1) transformations (phase rotations), in which the phase parameter depends on space-time. In order to accomplish this we include an interaction with a vector gauge boson which transforms under the local (gauge) transformation according to eq. (1.11).

- This interaction is encoded by replacing the derivative ∂_μ by the covariant derivative D_μ defined by eq. (1.22). $D_\mu \psi$ transforms under gauge transformations as $e^{-i\omega} D_\mu \psi$.
- The kinetic term for the gauge boson is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is proportional to the commutator $[D_\mu, D_\nu]$ and is invariant under gauge transformations.
- The gauge boson is massless, since a term proportional to $A_\mu A^\mu$ is *not* invariant under gauge transformations.
- The resulting Lagrangian is identical to that of QED.

2 Non-Abelian Gauge Theories

2.1 Non-Abelian Gauge Transformations

We now move on to apply the ideas of the previous lecture to the case where the transformations are "non-abelian", i.e. different elements of the group do not commute with each other. We shall restrict ourselves to the group $SU(n)$. This is the group of special, unitary $n \times n$ matrices. Special in this context means that the determinant is equal to 1. In order to specify an $SU(n)$ matrix completely we need $n^2 - 1$ real parameters. Indeed, we need $2n^2$ real parameters to determine an arbitrary complex $n \times n$ matrix. But there are n^2 constraints due to the unitary requirements and one additional constraint due to the requirement det = 1.

Thus, we write an arbitrary $SU(n)$ matrix as

$$e^{-i \sum_{a=1}^{n^2-1} \omega^a \mathbf{T}^a} \equiv e^{-i\omega^a \mathbf{T}^a} \tag{2.1}$$

where we adopted the summation convention. The ω^a, $a \in \{1 \dots n^2 - 1\}$ are the real parameters and the $\mathbf{T}^a$ are called the generators of the group.

> **Exercise 2.1**
> Show that the unitarity of the $SU(n)$ matrices entails hermiticity of the generators and that the requirement of det = 1 means that the generators have to be traceless.

In the case of $U(1)$ there was just one generator. Here we have $n^2 - 1$ generators $\mathbf{T}^a$. There is still some freedom left on how to normalize the generators. We will adopt the usual normalization convention

$$\mathrm{tr}(\mathbf{T}^a \mathbf{T}^b) = \frac{1}{2}\,\delta_{ab} \tag{2.2}$$

The reason we can always enforce eq. (2.2) is that $\mathrm{tr}(\mathbf{T}^a \mathbf{T}^b)$ is a real matrix symmetric in $a \leftrightarrow b$. Thus it can be diagonalized. If you have problems with getting on friendly terms with the concept of generators, for the moment you can think of them as traceless, hermitian $n \times n$ matrices (this is, however, not the complete picture).

The crucial new feature of the group $SU(n)$ is that two elements of $SU(n)$ generally do not commute, i.e.

$$e^{-i\omega_1^a \mathbf{T}^a} e^{-i\omega_2^b \mathbf{T}^b} \neq e^{-i\omega_2^b \mathbf{T}^b} e^{-i\omega_1^a \mathbf{T}^a} \tag{2.3}$$

(compare to eq. (1.3)). To put this somewhat more fancy, the group algebra is not trivial. For the commutator of two generators we have

$$[\mathbf{T}^a, \mathbf{T}^b] \equiv i f^{abc} \mathbf{T}^c \neq 0 \tag{2.4}$$

where we defined the structure constants of the group, f^{abc} and used the summation convention again. The structure constants are totally antisymmetric. This can be seen as follows: from eq. (2.4) it is obvious that $f^{abc} = -f^{bac}$. To convince us of the antisymmetry in the other indices as well, we note that multiplying eq. (2.4) by $\mathbf{T}^d$ and taking the trace, using eq. (2.2), we get $2if^{abd} = \mathrm{tr}(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^d) - \mathrm{tr}(\mathbf{T}^b\mathbf{T}^a\mathbf{T}^d) = \mathrm{tr}(\mathbf{T}^a\mathbf{T}^b\mathbf{T}^d) - \mathrm{tr}(\mathbf{T}^a\mathbf{T}^d\mathbf{T}^b)$.

Recalling eq. (1.2) we see that the fermion field, ψ_i, now carries an index i, which takes the value $i \in \{1 \dots n\}$. What we have actually done here is to assume that the fermions transform in the fundamental representation.[4] The conjugate[5] field is written $\overline{\psi}^i$.

The Lagrangian density for a free fermion field is

$$\mathcal{L} \;=\; \overline{\psi}^i \left(i\gamma^\mu\partial_\mu - m\right)\psi_i, \tag{2.5}$$

where the index i is summed over 1 to n. Eq. (2.5) is therefore a shorthand for

$$\mathcal{L} \;=\; \overline{\psi}^1 \left(i\gamma^\mu\partial_\mu - m\right)\psi_1 + \overline{\psi}^2 \left(i\gamma^\mu\partial_\mu - m\right)\psi_2 + \dots, \tag{2.6}$$

where $\psi_1, \psi_2 \dots$ are fermion fields that make up the multiplet ψ.

Under a $SU(n)$ gauge transformation the field ψ_i transforms as

$$\psi_i \to \left(e^{-i\omega^a\mathbf{T}^a}\right)_i^j \psi_j, \qquad \text{or} \qquad \psi \to \left(e^{-i\omega^a\mathbf{T}^a}\right)\psi \tag{2.7}$$

with i, j being the labels of the matrix $e^{i\omega^a\mathbf{T}^a}$. Once again, it is convenient to consider only infinitesimal transformations under which the field ψ_i changes by an infinitesimal amount $\delta\psi_i$, where

$$\delta\psi_i \;= -i\,\omega^a \left(\mathbf{T}^a\right)_i^j \psi_j, \qquad \text{or} \qquad \delta\psi = -i\,\omega^a\mathbf{T}^a\,\psi \tag{2.8}$$

and the conjugate field $\overline{\psi}^i$ changes by $\delta\overline{\psi}^i$, where

$$\delta\overline{\psi}^i \;= i\,\omega^a\overline{\psi}^j \left(\mathbf{T}^a\right)_j^i \qquad \text{or} \qquad \delta\overline{\psi} = i\,\omega^a\mathbf{T}^a\,\psi. \tag{2.9}$$

We see that these two changes cancel each other out in the Lagrangian eq. (2.5), provided that the parameters, ω^a are constant.

If we allow these parameters to depend on space-time, $\omega^a(x)$, then the Lagrangian density changes by $\delta\mathcal{L}$ under this "non abelian gauge transformation", where

$$\delta\mathcal{L} \;= \overline{\psi}^i \left(\mathbf{T}^a\right)_i^j \gamma^\mu \left(\partial_\mu\omega^a(x)\right)\psi_j. \tag{2.10}$$

This is exactly what we observed in the abelian case.

[4] This is not really required. It is related to the fact that we consider the generators to be $n \times n$ matrices. We could be more general and only require that the generators satisfy eq. (2.4).

[5] Note that the conjugate field has a superscript i because strictly speaking it transforms as the $\bar{n}$ representation of $SU(n)$.

2.2 Non-Abelian Gauge Fields

The local gauge symmetry can once again be restored by introducing interactions with vector (spin-one) gauge bosons. In this case we need $n^2 - 1$ such gauge bosons, A^a_μ, one for each generator of $SU(n)$. Under an infinitesimal gauge transformation these gauge bosons transform as

$$\delta A^a_\mu(x) \; = -f^{abc} A^b_\mu(x)\,\omega^c(x) + \frac{1}{g}[\partial_\mu \omega^a(x)]. \tag{2.11}$$

The first term on the right hand side of eq. (2.11) is the transformation similar to eq. (2.9) and is even present for constant ω^a. The second term, (which is non-linear in the field A^a_μ) is an extra term which needs to be added for the case of space-time dependent ω^a.

The interaction with these gauge bosons is again encoded by replacing the ordinary partial derivative in the Lagrangian density (2.5) with a covariant derivative, which in this case is a $n \times n$ matrix defined by

$$\mathbf{D}_\mu \; = \; \left(\partial_\mu \mathbf{I} + i\,g\,\mathbf{T}^a A^a_\mu\right), \tag{2.12}$$

where $\mathbf{I}$ is the unit matrix. The Lagrangian density thus becomes

$$\mathcal{L} \; = \; \overline{\psi}^i \; (i\gamma^\mu \mathbf{D}_\mu - m\mathbf{I})_i^j \, \psi_j, \tag{2.13}$$

The quantity $\mathbf{D}_\mu \psi$ is not invariant under gauge transformations, but using eqs. (2.8) and (2.11) and the commutation relations eq. (2.4) we obtain the change of $\mathbf{D}_\mu \psi$ under an infinitesimal gauge transformation

$$\delta\left(\mathbf{D}_\mu \psi\right) \; = -i\,\omega^a \mathbf{T}^a \mathbf{D}_\mu \psi. \tag{2.14}$$

Together with eq. (2.9) this tell us that the new Lagrangian density eq. (2.13) is invariant under local $SU(n)$ gauge transformations.

We can express the transformation rule for $\mathbf{D}_\mu \psi$ in terms of a transformation rule for the matrix $\mathbf{D}_\mu$ as

$$\delta \mathbf{D}_\mu \; = -i\,\left[\omega^a \mathbf{T}^a, \mathbf{D}_\mu\right] \tag{2.15}$$

Exercise 2.2
Use eqs. (2.8) and (2.11) to verify eq. (2.14), and show also that this would *not* work if the ordinary partial derivative were used in place of the covariant derivative.

Exercise 2.3
Show that eq. (2.15) together with eq. (2.8) give eq. (2.14). (you do *not* need to derive eq. (2.15)).

The kinetic term for the gauge bosons is again constructed from the field strengths $F^a_{\mu\nu}$ which are defined from the commutator of two covariant derivatives:

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}\,[\mathbf{D}_\mu, \mathbf{D}_\nu]\,. \tag{2.16}$$

where the matrix $\mathbf{F}_{\mu\nu}$ is given by

$$\mathbf{F}_{\mu\nu} = \mathbf{T}^a F^a_{\mu\nu}, \tag{2.17}$$

This gives us

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}\, A^b_\mu A^c_\nu \tag{2.18}$$

From eq. (2.14) we obtain the change in $\mathbf{F}_{\mu\nu}$ under an infinitesimal gauge transformation as

$$\delta\mathbf{F}_{\mu\nu} = -i\omega^a\,[\mathbf{T}^a, \mathbf{F}_{\mu\nu}] \tag{2.19}$$

which leads to

$$\delta F^a_{\mu\nu} = -f^{abc}\, F^b_{\mu\nu}\, \omega^c. \tag{2.20}$$

The gauge invariant term which contains the kinetic term for the gauge bosons is therefore $-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}$, where a summation over the index a is implied.

In sharp contrast with the abelian case, this term does not only contain the terms which are quadratic in the derivatives of the gauge boson fields, but also the terms

$$g\, f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu - \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu. \tag{2.21}$$

This means that there is a very important difference between abelian and non-abelian gauge theories. For non-abelian gauge theories the gauge bosons interact with each other via both three-point and four-point interaction terms. The three point interaction term contains a derivative, which means that the Feynman rule for the three-point vertex involves the momenta of the particles going into the vertex. We shall write down the Feynman rules in detail later.

Once again, a mass term for the gauge bosons is forbidden, since a term proportional to $A^a_\mu A^{a\,\mu}$ is *not* invariant under gauge transformations.

2.3 Gauge Fixing

As in the case of QED, we need to add a gauge fixing term in order to be able to derive a propagator for the gauge bosons. In the Feynman gauge this means adding the term $-\frac{1}{2}(\partial^\mu A^a_\mu)^2$ to the Lagrangian density and the propagator (in momentum space) becomes

$$-i\,\delta_{ab}\,\frac{g_{\mu\nu}}{p^2}.$$

There is one unfortunate complication, which is included briefly here for the sake of completeness, although one only needs to know about it for the purpose of performing higher loop calculations with non-abelian gauge theories.

If one goes through the formalism of gauge fixing carefully, it turns out that at higher orders this leads to extra loop diagrams. These diagrams involve additional particles that are mathematically equivalent to interacting scalar particles and are known as a "Faddeev-Popov ghost". For each gauge field there is such a ghost field. These are *not* to be interpreted as physical scalar particles which could in principle be observed experimentally, but merely as part of the gauge-fixing programme. For this reason they are referred to as 'ghosts". Furthermore they have two peculiarities

1. They only occur inside loops. This is because they are not really particles and cannot occur in initial or final states, but are introduced to clean up a difficulty that arises in the gauge-fixing mechanism.

2. They behave like fermions even though they are scalars (spin zero). This means that we need to count a minus sign for each loop of Faddeev-Popov ghosts in any Feynman diagram.

We shall display the Feynman rules for these "ghosts" later.

Thus, for example, the Feynman diagrams which contribute to the one-loop corrections to the gauge boson propagator are

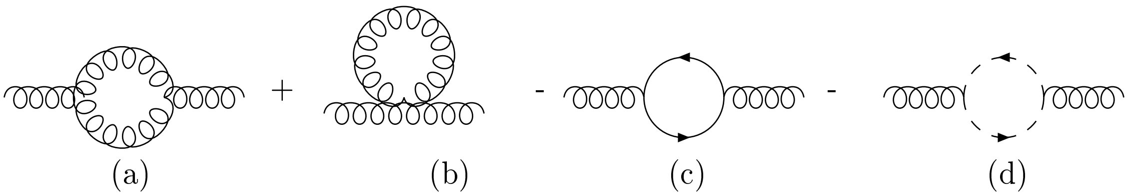


Diagram (a) involves the three-point interaction between the gauge bosons, diagram (b) involves the four-point interaction between the gauge bosons, diagram (c) involves a loop of fermions, and diagram (d) is the extra diagram involving the Faddeev-Popov ghosts. Note that both diagrams (c) and (d) have a minus sign in front of them because both the fermions and Faddeev-Popov ghosts obey Fermi statistics.

2.4 The Lagrangian for a General Non-Abelian Gauge Theory

Let us summarize what we have found so far: Consider a group gauge group, $\mathcal{G}$ of "dimension" N (for $SU(n): \ N \equiv n^2-1$), whose N generators, $\mathbf{T}^a$, obey the commutation relations $\left[\mathbf{T}^a, \mathbf{T}^b\right] = i f_{abc}\mathbf{T}^c$, where f_{abc} are called the "structure constants" of the group.

The Lagrangian density for a gauge theory with this group, with a fermion multiplet ψ_i is given (in Feynman gauge) by

$$\mathcal{L} \ = \ -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + i\overline{\psi}\left(\gamma^\mu\mathbf{D}_\mu - m\mathbf{I}\right)\psi - \frac{1}{2}(\partial^\mu A^a_\mu)^2 + \mathcal{L}_{FP} \qquad (2.22)$$

where

$$F^a_{\mu\nu} \ = \ \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}A^b_\mu A^c_\nu, \qquad (2.23)$$
$$\mathbf{D}_\mu \ = \ \partial_\mu\mathbf{I} + i\,g\,\mathbf{T}^a A^a_\mu \qquad (2.24)$$

and

$$\mathcal{L}_{FP} \ = \ -\xi^a\partial^\mu\partial_\mu\eta^a + g\, f_{acb}\xi^a A^c_\mu(\partial^\mu\eta^b) \qquad (2.25)$$

Under an infinitesimal gauge transformation, the N gauge bosons, A^a_μ change by an amount that contains a term which is not linear in A^a_μ:

$$\delta A^a_\mu(x) \ = -f^{abc}A^b_\mu(x)\omega^c(x) + \frac{1}{g}\partial_\mu\omega^a(x), \qquad (2.26)$$

whereas the field strengths $F^a_{\mu\nu}$ transform by a change

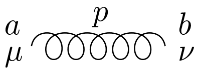
$$\delta F^a_{\mu\nu}(x) \ = -f^{abc}\,F^b_{\mu\nu}(x)\,\omega^c. \qquad (2.27)$$

In other words they transform as the "adjoint" representation of the group (which has as many components as there are generators). This means that the quantity $F^a_{\mu\nu}F^{a\,\mu\nu}$ (summation over a implied) is invariant under gauge transformations.

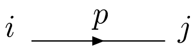
2.5 Feynman Rules

The Feynman rules for such a gauge theory can directly be read off from the Lagrangian. As mentioned previously, the propagators are obtained by taking all terms bilinear in the field and inverting the corresponding operator (and multiplying by i). The rules for the vertices are obtained by simply taking (i times) the factor multiplying the corresponding term in the Lagrangian. The explicit rules are given by:

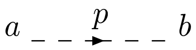
Propagators:



Gluon: $-i\,\delta_{ab}\,g_{\mu\nu}/p^2$

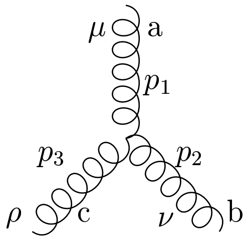


Fermion: $i\,\delta_{ij}(\gamma^\mu p_\mu + m)/(p^2 - m^2)$

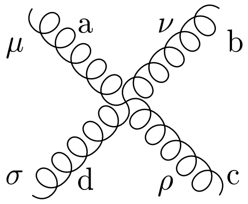


Faddeev-Popov ghost: $i\,\delta_{ab}/p^2$

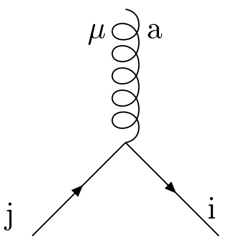
Vertices: (all momenta are flowing into the vertex).



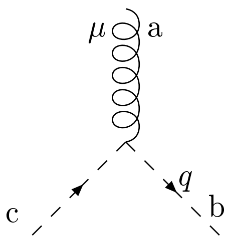
$$-g\,f^{abc}\left(g_{\mu\nu}\,(p_1-p_2)_\rho + g_{\nu\rho}\,(p_2-p_3)_\mu + g_{\rho\mu}\,(p_3-p_1)_\nu\right)$$



$$-i\,g^2 f^{eab} f^{ecd}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$
$$-i\,g^2 f^{eac} f^{ebd}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho})$$
$$-i\,g^2 f^{ead} f^{ebc}\,(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})$$



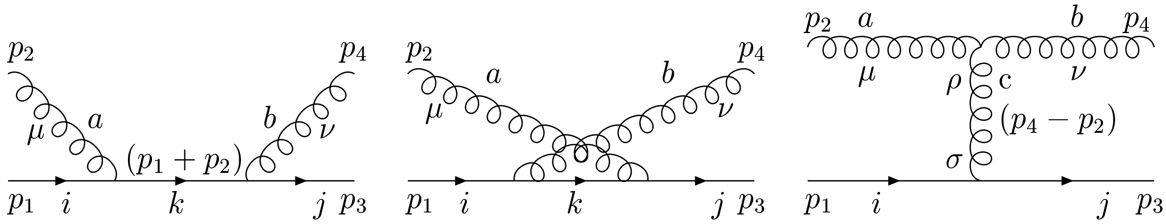
$$-i\,g\,\gamma^\mu\,(T^a)_{ij}$$



$$g\,f^{abc}\,q_\mu$$

2.6 An Example

As an example of the application of these Feynman rules, we consider the process of Compton scattering, but this time for the scattering of non-abelian gauge bosons and fermions, rather than photons. We need to calculate the amplitude for a gauge boson of momentum p_2 and gauge label a to scatter fermion of momentum p_1 and gauge label i producing a fermion of momentum p_3 and gauge label j and a gauge boson of momentum p_4 and gauge label b. Note that $i, j \in \{1 \dots n\}$ whereas $a, b \in \{1 \dots n^2 - 1\}$. In addition to the two Feynman diagrams one gets in the QED case there is a third diagram involving the self-interaction of the gauge bosons.



(a) (b) (c)

We will assume that the fermions are massless (i.e. that we are at sufficiently high energies that we may neglect their masses), and work in terms of the Mandelstam variables

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2
\end{aligned}
$$

The polarizations are accounted for by contracting the amplitude obtained for the above diagrams with the polarization vectors $\epsilon^\mu(\lambda_2)$ and $\epsilon^\nu(\lambda_4)$. Each diagram consists of two vertices and a propagator and so their contributions can be read off from the Feynman rules.

For diagram (a) we get

$$
\begin{aligned}
& \epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\, g\, \gamma^\nu(\mathbf{T}^b)^k_j\right)\left(i\,\frac{\gamma\cdot(p_2+p_2)}{s}\right)\left(-i\, g\, \gamma^\mu(\mathbf{T}^a)^i_k\right)u_i(p_1) \\
= & -i\,\frac{g^3}{s}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1+p_2)\gamma^\mu\right)\left(\mathbf{T}^b\mathbf{T}^a\right)u(p_1).
\end{aligned}
$$

For diagram (b) we get

$$
\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\, g\, \gamma^\mu(\mathbf{T}^a)^k_j\right)\left(i\,\frac{\gamma\cdot(p_1-p_4)}{u}\right)\left(-i\, g\, \gamma^\nu(\mathbf{T}^b)^i_k\right)u_i(p_1)
$$

$$= -i\,\frac{g^3}{u}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1-p_4)\gamma^\mu\right)\left(\mathbf{T}^a\mathbf{T}^b\right)u(p_1).$$

Note here that the order of the $\mathbf{T}$ matrices is the other way around from diagram (a).

Diagram (c) involves the three-point gauge-boson self-coupling. Since the Feynman rule for this vertex is given with incoming momenta, it is useful to replace the outgoing gauge-boson momentum p_4 by $-p_4$ and understand this to be an incoming momentum. Note that the internal gauge-boson line carries momentum $p_4 - p_2$ coming into the vertex. The three incoming momenta that are to be substituted into the Feynman rule for the vertex are therefore p_2, $-p_4$, $p_4 - p_2$. The vertex thus becomes

$$-g\,f_{abc}\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right)$$

and the diagram gives

$$\begin{aligned}
&\epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma_\sigma(\mathbf{T}^c)^i_j\right)u_i(p_1)\left(-i\frac{g^{\rho\sigma}}{t}\right)\\
&\times\;(-g\,f_{abc})\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right)\\
= &-i\frac{g^3}{t}\epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_4)\overline{u}\,(p_3)\left[\mathbf{T}^a,\mathbf{T}^b\right]\gamma^\rho u(p_1)\left(g_{\mu\nu}\,(p_2+p_4)_\rho - 2(p_4)_\mu g_{\nu\rho} - 2(p_2)_\nu g_{\mu\rho}\right),
\end{aligned}$$

where in the last step we have used the fact that the polarization vectors are transverse so that $p_2\cdot\epsilon(\lambda_2)=0$ and $p_4\cdot\epsilon(\lambda_4)=0$ and the commutation relations eq. (2.4).

Exercise 2.4

Draw all the Feynman diagrams for the tree level amplitude for two gauge bosons with momenta p_1 and p_2 to scatter into two gauge bosons with momenta q_1 and q_2. Label the momenta of the external gauge boson lines.

2.7 Summary

- A non-abelian gauge theory is one in which the Lagrangian is invariant under local transformations of a non-abelian group.
- This invariance is achieved by introducing a gauge boson for each generator of the group. The partial derivative in the Lagrangian for the fermion field is replaced by a covariant derivative defined in eq. (2.24).
- The gauge bosons transform under infinitesimal gauge transformations in a non-linear way given by eq. (2.26).

- The field strengths, $F^a_{\mu\nu}$ are obtained from the commutator of two covariant derivatives and are given by eq. (2.23). They transform as the adjoint representation under gauge transformations such that the quantity $F^a_{\mu\nu}F^{a\,\mu\nu}$ is invariant.

- $F^a_{\mu\nu}F^{a\,\mu\nu}$ contains terms which are cubic and quartic in the gauge bosons, indicating that these gauge bosons interact with each other.

- The gauge fixing mechanism leads to the introduction of Faddeev-Popov ghosts which are scalar particles that occur only inside loops and obey Fermi statistics.

3 Quantum Chromodynamics

Quantum Chromodynamics (QCD) is the theory of the strong interactions. It is nothing but a non-abelian gauge theory with the group $SU(3)$. Thus, the quarks are described by a field ψ_i where i runs from 1 to 3. The quantum number associated with the label i is called color. The eight gauge bosons which have to be introduced in order to preserve local gauge invariance are the eight 'gluons'. These are taken to be the carriers which mediate the strong interactions in exactly the same way that photons are the carriers which mediate the electromagnetic interactions.

The Feynman rules for QCD are therefore simply the Feynman rules listed in the previous lecture, with the gauge coupling constant, g, taken to be the strong coupling, g_s, (more about this later), the generators $\mathbf{T}^a$ taken to be the eight generators of $SU(3)$ in the triplet representation, and f^{abc}, $a, b, c, = 1 \dots 8$ are the structure constants of $SU(3)$ (you look these up in a book whenever you need them).

Thus we now have a quantum field theory which can be used to describe the strong interactions.

3.1 Running Coupling

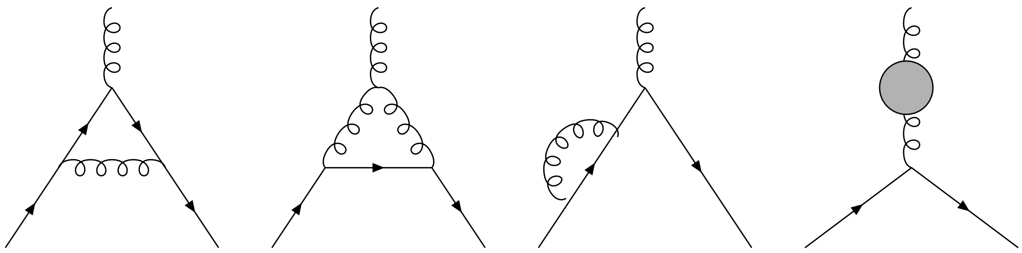
The coupling for the strong interactions is the QCD gauge coupling, g_s. We usually work in terms of α_s defined as

$$\alpha_s = \frac{g_s^2}{4\pi}. \tag{3.1}$$

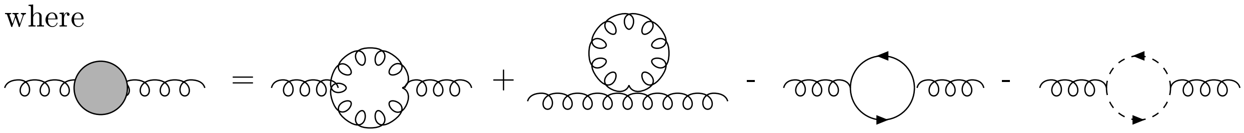
Since the interactions are strong, we would expect α_s to be too large to perform reliable calculations in perturbation theory. On the other hand the Feynman rules are only useful within the context of perturbation theory.

This difficulty is resolved when we understand that 'coupling constants' are not constant at all. The electromagnetic fine structure constant, α, only has the value $1/137$ at energies which are not large compared with the electron mass. At higher energies it is larger than this. For example, at LEP energies it takes a value closer to $1/128$. On the other hand, it turns out that for non-abelian gauge theories the coupling *decreases* as the energy increases.

To see how this works within the context of QCD, we note that when we perform higher order perturbative calculations there are loop diagrams which have the effect of dressing the couplings. For example, the one-loop diagrams which dress the coupling between a quark and a gluon are:



where



are the diagrams needed to calculate the one-loop corrections to the gluon propagator.

These diagrams contain UV divergences and need to be renormalized by subtracting at some renormalization scale μ. This scale then appears inside a logarithm for the renormalized quantities. This means that if the square-momenta of all the external particles coming into the vertex are of order Q^2, where $Q \gg \mu$, then the above diagrams give rise to a correction which contains a logarithm of the ratio Q^2/μ^2:

$$-\alpha_s^2\,\beta_0\,\ln\left(Q^2/\mu^2\right). \tag{3.2}$$

This correction is interpreted as the correction to the effective QCD coupling, $\alpha_s(Q^2)$, at momentum scale Q. i.e

$$\alpha_s(Q^2) \;= \alpha_s(\mu^2) \;-\; \alpha_s(\mu^2)^2\,\beta_0\,\ln\left(Q^2/\mu^2\right) \;+\; \ldots \tag{3.3}$$

β_0 is calculated to be

$$\beta_0 \;=\; \frac{11\,N_c \;-\; 2\,n_f}{12\,\pi}, \tag{3.4}$$

where N_c is the number of colors (=3), n_f is the number of active flavors, i.e. the number of flavors whose mass threshold is below the momentum scale, Q. Note that β_0 is *positive*, which means that the coefficient in front of the logarithm in eq. (3.3) is *negative*, so that the effective coupling *decreases* as the momentum scale is increased.

A more precise analysis shows that the effective coupling obeys the differential equation

$$\frac{\partial\,\alpha_s(Q^2)}{\partial\,\ln(Q^2)} \;=\; \beta\left(\alpha_s(Q^2)\right) \tag{3.5}$$

where β has a perturbative expansion

$$\beta(\alpha) \;=\; -\beta_0\,\alpha^2 + \mathcal{O}(\alpha^3) + \cdots \tag{3.6}$$

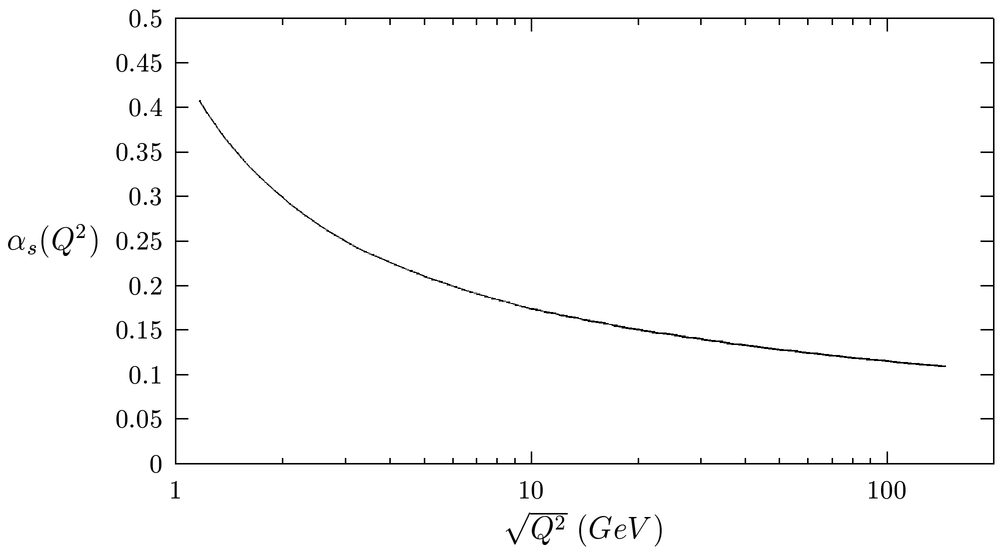


Figure 3.1: The running of $\alpha_s(Q^2)$ with β taken to two loops.

In order to solve this differential equation we need a boundary value. Nowadays this is taken to be the measured value of the coupling at the Z boson mass[6] (= 91 GeV), which is measured to be

$$\alpha_s(M_Z^2) \;=\; 0.118 \pm 0.002 \tag{3.7}$$

This is one of the free parameters of the Standard Model.

The running of $\alpha_s(Q^2)$ is shown in the figure 3.1. We can see that for momentum scales above about 2 GeV the coupling is less than 0.3 so that one can hope to carry out reliable perturbative calculations for QCD processes with energy scales larger than this.

Gauge invariance requires that the gauge coupling for the interaction between gluons must be exactly the same as the gauge coupling for the interaction between quarks and gluons. The β-function could therefore have been calculated from the higher order corrections to the three-gluon (or four-gluon) vertex and must yield the same result, despite the fact that it is calculated from a completely different set of diagrams.

[6]Previously the solution to eq. (3.5) (to leading order) was written as $\alpha_s(Q^2) = 4\pi/\beta_0 \ln(Q^2/\Lambda^2_{QCD})$ and the scale Λ_{QCD} was used as the standard parameter which sets the scale for the magnitude of the strong coupling. This turns out to be rather inconvenient since it needs to be adjusted every time higher order corrections are taken into consideration and the number of active flavors has to be specified. This parametrization is now hardly ever used.

Exercise 3.1
Draw the Feynman diagrams needed for the calculation of the one-loop correction to the triple gluon coupling (don't forget the Faddeev-Popov ghost loops).

Exercise 3.2
Solve equation (3.5) using β to leading order only, and calculate the value of α_s at a momentum scale of 10 GeV. Use the value at M_z given by eq. (3.7). Calculate also the error in α_s at 10 GeV.

3.2 Quark (and Gluon) Confinement

This argument can be inverted to provide an answer to the question 'why have we never seen quarks or gluons in a laboratory ? '.

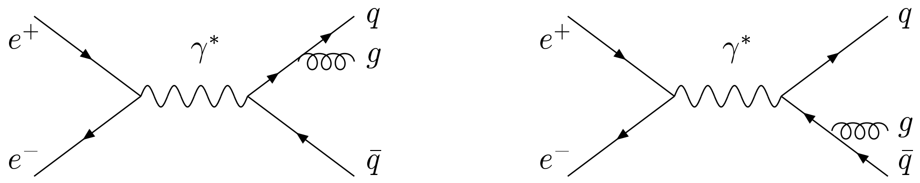
Asymptotic freedom, which tells us that the effective coupling between quarks becomes weaker as we go to short distances (this is equivalent to going to high energies) implies, conversely, that effective couplings grow as we go to large distances. Therefore, the complicated system of gluon exchanges, which leads to the binding of quarks (and antiquarks) inside hadrons, leads to a stronger and stronger binding as we attempt to pull the quarks apart. This means that we can never isolate a quark (or a gluon) at large distances since we require more and more energy to overcome the binding as the distance between the quarks grows.

The upshot of this is that the only free particles which can be observed at macroscopic distances from each other are color singlets. This mechanism is known as "quark confinement". The details of how it works are not fully understood. Nevertheless the argument presented here is suggestive of such confinement and at the level of non-perturbative field theory, lattice calculations have confirmed that for non-abelian gauge theories the binding energy does indeed grow as the distance between quarks increases.

Thus we have two different pictures of the world. At sufficiently short distances, which can be probed at sufficiently large energies, we can consider quarks and gluons interacting with each other. We can perform calculations of the scattering cross sections between quarks and gluons (called the "hard cross section") at short distances because the running coupling is sufficiently small that we can rely on perturbation theory.

On the other hand, before we can make a direct comparison with what is observed in accelerator experiments, we need to take into account the fact that these quarks and gluons bind into color singlet hadrons and it is only these color singlet states that are observed. The mechanism for such binding is beyond the scope of perturbation theory and is not understood in detail. Nevertheless several Monte Carlo programs have been developed which simulate this binding in such a way that the results of the short-distance perturbative calculations at the level of quarks and gluons can be confronted with experiment in a successful way.

Thus, for example, if we wish to calculate the cross section for electron-positron annihilation into three jets (at high energies), we first calculate, in perturbation theory, the process for electron plus positron to annihilate into a virtual photon which then decays into a quark, and antiquark and a gluon. The two Feynman diagrams for this process are:



However, before we can compare the results of this perturbative calculation with experimental data on three jets of observed hadrons, we need to perform a convolution of this calculated cross section with a Monte Carlo simulation that accounts for the way in which the final state partons (quarks and gluons) bind with other quarks and gluons to produce observed hadrons. It is only after such a convolution has been performed that one can get a reliable comparison of the calculated cross section with data.

Likewise, if we want to calculate cross sections for initial state hadrons we need to account for the probability of finding a particular quark or gluon inside an initial hadron with a given fraction of the initial hadron's momentum (these are called "parton distribution functions").

Exercise 3.3
Draw the (tree level) Feynman diagrams for the process $e^+ + e^- \to$ 4 jets. Consider only one photon exchange plus the QCD contributions only (do not include $Z-$boson exchange or $W-W$ production.)

3.3 $\theta-$ Parameter

There is one more gauge invariant term that can be written down:

$$\mathcal{L}_\theta = \theta \frac{\alpha_s}{8\pi} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}, \tag{3.8}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor (in four dimensions) and the factor 8π is purely conventional. Since we should work with the most general gauge invariant Lagrangian there is no reason to omit this term. However, adding this term to the Lagrangian leads to a problem, the strong CP problem.

To understand the nature of the problem, we first convince ourselves that this term violates CP. In QED we would have

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \;=\; \mathbf{E} \cdot \mathbf{B}, \tag{3.9}$$

and for QCD we have a similar expression except that $\mathbf{E}^a$ and $\mathbf{B}^a$ carry a color index – they are known the chromoelectric and chromomagnetic fields. Under charge conjugation both the electric and magnetic field change sign. But under parity the electric field, which is a proper vector, changes sign, whereas the magnetic field, which is a polar vector, does not change sign. Thus we see that the term $\mathbf{E} \cdot \mathbf{B}$ is odd under CP.

For this reason, the parameter θ in front of this term must be exceedingly small in order not to give rise to strong interaction contributions to CP violating quantities such as the electric dipole moment of the neutron. The current experimental limits on this dipole moment tell us that $\theta < 10^{-9}$. Thus we are tempted to think that is zero. Nevertheless, strictly speaking θ is a free parameter of QCD, and is often considered to be the nineteenth free parameter of the Standard Model.

Of course we simply could set θ to zero (or a very small number) and be happy with it. However, whenever a free parameter is zero or extremely small, we would like to understand the reason. The fact that we do not know why this term is absent (or so small) is the strong CP problem.

There are several possible solutions to the strong CP problem that offer explanations why this term is absent (or small). But none of these solutions has been confirmed yet and the problem is still unresolved.

Another question is why this is not a problem in QED. In fact a term like eq. (3.8) can also be written down in QED. A thorough discussion of this point is beyond the scope of this lecture. Suffice to say that this term can be written (in QED and QCD) as a total divergence. So it seems that it can be eliminated from the Lagrangian altogether. However, in QCD (but not in QED) there are non-perturbative effects (related to so called instantons) which prevent us from neglecting the θ-term.

3.4 Summary

- Quarks transform as a triplet representation of color $SU(3)$ (each quark can have one of three colors.)

- The eight gauge bosons of QCD are the gluons which are the carriers that mediate the strong interactions.

- The coupling of quarks to gluons (and gluons to each other) decreases as the energy scale increases. Therefore, at high energies one can perform reliable perturbative calculations for strong interacting processes.

- As the distance between quarks increases the binding increases, such that it is impossible to isolate individual quarks or gluons. The only observable particles are color singlet hadrons. Perturbative calculations performed at the quark and gluon level must be modified by accounting for the recombination of final state quarks and gluons into observed hadrons as well as the probability of finding these quarks and gluons inside the initial state hadrons.

- QCD admits a gauge invariant strong CP violating term with a coefficient θ. This parameter is known to be very small from limits on CP violating phenomena such as the electric dipole moment of the neutron.

4 Spontaneous Symmetry Breaking

We have seen that in an unbroken gauge theory the gauge bosons must be massless. This is exactly what we want for QED (massless photon) and QCD (massless gluons). However, if we wish to extend the ideas of describing interactions by a gauge theory to the weak interactions, the symmetry must somehow be broken since the carriers of the weak interactions (W and Z bosons) are massive (weak interactions are very short range). We could simply break the symmetry by hand by adding a mass term for the gauge bosons, which we know violates the gauge symmetry. However, this would destroy renormalizability of our theory. There is a far more elegant way of breaking the symmetry which is called "spontaneous symmetry breaking". In this scenario, the Lagrangian maintains its symmetry under a set of local gauge transformations. On the other hand, the lowest energy state, which we interpret as the vacuum state, is *not* a singlet of the gauge symmetry. There is an infinite number of states each with the same ground-state energy and nature chooses one of these states as the 'true' vacuum.

4.1 Massive Gauge Bosons and Renormalizability

In this subsection we will convince ourselves that simply adding a mass term for the gauge bosons by hand will destroy the renormalizability of the theory. Of course, we can not give a rigorous argument since this would be well beyond the scope of these notes. However, the argument given here is good enough to illustrate the crucial difference between introducing mass terms for the gauge bosons in a brute force way and introducing them via spontaneous symmetry breaking.

Higher order (loop) corrections generate ultraviolet divergences. In a renormalizable theory, these divergences can be absorbed into the parameters of the theory and in this way can be 'hidden'. As we go to higher orders we need to absorb more and more terms into these parameters, but there are only as many divergent quantities as there are parameters.

There are restrictions on the allowed interaction terms. Furthermore all the propagators have to decrease like $1/p^2$ as the momentum $p \to \infty$. Note that this is how the massless gauge-boson propagator eq. (1.21) behaves. If these conditions are not fulfilled then the theory generates more and more divergent terms as one calculates to higher orders and it is not possible to absorb these divergences into the parameters of the theory. Such theories are said to be "non-renormalizable".

Now we can convince ourselves that simply adding a mass term $M^2\, A_\mu A^\mu$ to the Lagrangian given in eq. (2.22) will lead to a non-renormalizable theory. To start with we note that such a term will modify the propagator. Collecting all terms bilinear in the gauge fields in momentum space and Feynman gauge we get

$$\frac{1}{2}A_\mu\left(-g^{\mu\nu}(p^2-M^2)+p^\mu p^\nu\right)A_\nu \tag{4.1}$$

We have to invert this operator to get the propagator which now takes the form

$$\frac{1}{p^2-M^2}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M^2}\right) \tag{4.2}$$

Note that this propagator, eq. (4.2), has a much worse ultraviolet behavior in that it behaves like p^0 for $p \to \infty$. Thus, it is clear that the ultraviolet properties of a theory with a propagator as given in eq. (4.2) are worse that for a theory with a propagator as given in eq. (1.21). According to our discussion at the beginning of this subsection we conclude that without the explicit mass term $M^2\, A_\mu A^\mu$ the theory is renormalizable whereas with this term it is not. In fact, it is precisely the gauge symmetry that ensures renormalizability. Breaking this symmetry results in the loss of renormalizability.

The crucial question now is whether there is a way to break the gauge symmetry in a somewhat more subtle way such that we still can give the gauge bosons a mass but retain renormalizability. The answer is that breaking the gauge symmetry spontaneously does precisely this.

4.2 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is a phenomenon that is by far not restricted to gauge symmetries. It is a subtle way to break a symmetry by still requiring that the Lagrangian remains invariant under the symmetry transformation. However, the ground state of the symmetry is *not* invariant, i.e. *not* a singlet under a symmetry transformation.

In order to illustrate the idea of spontaneous symmetry breaking consider a pen that is completely symmetric with respect to rotations around its axis. If we put this pen on a table and start to press on it with a force precisely along the axis we have a perfectly symmetric situation. This corresponds to a Lagrangian that is symmetric (under rotations around the axis of the pen in this case). However, if we increase the force, at some point the pen will bend (and eventually break). The question then is in which direction it will bend. Of course we do not know, since all directions are equal. But the pen will pick one and by doing so it will break the rotational symmetry. This is the spontaneous symmetry breaking.

A better example can be given by looking at point mass in a potential

$$V(\vec{r}) \;=\; \mu^2\vec{r}\cdot\vec{r}+\lambda(\vec{r}\cdot\vec{r})^2. \tag{4.3}$$

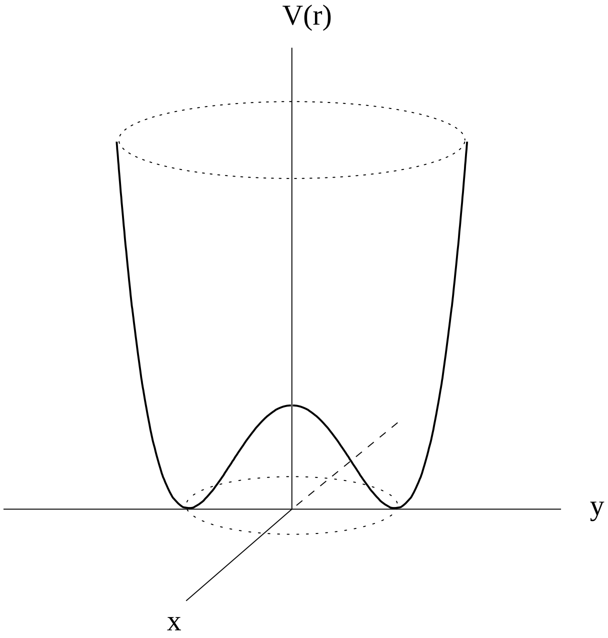


Figure 4.1: A potential that leads to spontaneous symmetry breaking

This potential is symmetric under rotations and we assume $\lambda > 0$. For $\mu^2 > 0$ the potential has a minimum at $\vec{r} = 0$, thus the point mass will simply fall to this point. The situation is more interesting, if $\mu^2 < 0$. For 2-dimensional $\vec{r}$ the potential is shown in Fig. 4.1. If the point mass sits at $\vec{r} = 0$ the system is not in the ground state but the situation is completely symmetric. In order to reach the ground state, the symmetry has to be broken, i.e. if the point mass wants to roll down, it has to decide in which direction. Any directions is equally good, but one has to be picked. This is exactly what spontaneous symmetry breaking means. The Lagrangian (here the potential) is symmetric (here under rotations around the z-axis), but the ground state (here the position of the point mass once it rolled down) is not.

Let us formulate this in a slightly more mathematical way. Since we will apply these ideas to gauge symmetries, we will now return to gauge symmetries. We denote the ground state by $|0\rangle$. A spontaneously broken gauge theory is a theory whose Lagrangian is invariant under gauge transformations, which is exactly what we have done in sections 1 and 2. The new feature in a spontaneously broken theory is that the ground state is not invariant under gauge transformations. This means

$$e^{-i\omega^a \mathbf{T}^a} |0\rangle \neq |0\rangle \tag{4.4}$$

which entails

$$\mathbf{T}^a |0\rangle \neq 0 \qquad \text{for some} \quad a \tag{4.5}$$

Eq. (4.5) follows from eq. (4.4) upon expansion in ω^a. Thus, the theory is spontaneously broken if there exists at least one generator that does not annihilate the vacuum.

In the next section we will apply these ideas in more detail to gauge symmetries and we will see that indeed, this way of breaking the gauge symmetry has all the desired features.

4.3 Summary

- A symmetry is said to be spontaneously broken if the Lagrangian is still invariant under the symmetry transformations, but the ground state is not.
- The symmetry associated with the generator $\mathbf{T}$ is spontaneously broken if $\mathbf{T}|0\rangle \neq 0$.

5 The Higgs Mechanism

In this section we will illustrate how to break a gauge symmetry spontaneously, how this introduces a mass for the gauge boson without destroying renormalizability and what all this has to do with a scalar (i.e. Higgs) field. For simplicity, we will consider the abelian case that is a $U(1)$ gauge theory. In the case of the Standard Model it will be a non-abelian gauge theory that is spontaneously broken but all the important ideas can simply be translated from the $U(1)$ case considered here.

5.1 The Abelian Model

We start by considering a gauged complex scalar field theory with a mass term and a quartic self-interaction. The Lagrangian density for such a theory may be written

$$\mathcal{L} \;=\; (D_\mu \Phi)^+ D^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\Phi), \tag{5.1}$$

where the potential $V(\Phi)$, is given by

$$V(\Phi) \;=\; \mu^2 \Phi^+ \Phi + \lambda \left|\Phi^+ \Phi\right|^2. \tag{5.2}$$

and the covariant derivative D_μ and the field-strength tensor $F_{\mu\nu}$ are given in eqs. (1.22) and (1.12) respectively. This Lagrangian is invariant under $U(1)$ gauge transformations

$$\Phi \;\to\; e^{-i\omega(x)} \Phi. \tag{5.3}$$

Provided μ^2 is positive this potential has a minimum at $\Phi = 0$. We call the $\Phi = 0$ state the vacuum and expand Φ in terms of creation and annihilation operators that populate the higher energy states. In terms of a quantum field theory, where Φ is an operator, the precise statement is that the operator Φ has zero vacuum expectation value, i.e. $\langle 0|\Phi|0\rangle = 0$.

In section 4 we have seen that reversing the sign of μ^2 results in spontaneous symmetry breaking. Let's pursue this idea so that the potential becomes

$$V(\Phi) \;=\; -\mu^2 \Phi^+ \Phi + \lambda \left|\Phi^+ \Phi\right|^2. \tag{5.4}$$

with $\mu^2 > 0$. We see that this potential no longer has a minimum at $\Phi = 0$, but a *maximum*. The minimum occurs at

$$\Phi = e^{i\theta} \sqrt{\frac{\mu^2}{2\lambda}} \equiv e^{i\theta} \frac{v}{\sqrt{2}} \tag{5.5}$$

where θ takes any value from 0 to 2π. There is an infinite number of states each with the same lowest energy, i.e. we have a degenerate vacuum. The symmetry breaking occurs in the choice made for the value of θ which represents the true vacuum. For convenience we shall choose $\theta = 0$ to be this vacuum. Such a choice constitutes a spontaneous breaking of the $U(1)$ invariance, since a $U(1)$ transformation takes us to a different lowest energy state. In other words the vacuum breaks $U(1)$ invariance. In quantum field theory we say that the field Φ has a non-zero vacuum expectation value

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}}. \tag{5.6}$$

But this means that there are 'excitations' with zero energy, that take us from the vacuum to one of the other states with the same energy. The only particles which can have zero energy are massless particles (with zero momentum). We therefore expect a massless particle in such a theory.

To see that we do indeed get a massless particle, we expand Φ around its vacuum expectation value as

$$\Phi = \frac{1}{\sqrt{2}} \left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi \right). \tag{5.7}$$

The fields H and ϕ have zero vacuum expectation value and it is these fields that are expanded in terms of creation and annihilation operators of the particles that populate the excited states. Of course, it is the H-field that corresponds to the Higgs field.

We now want to write the Lagrangian in terms of the H and ϕ fields. In order to get the potential we insert eq. (5.7) into eq. (5.4) and find

$$V = \mu^2 H^2 + \mu\sqrt{\lambda} \left(H^3 + \phi^2 H \right) + \frac{\lambda}{4} \left(H^4 + \phi^4 + 2H^2 \phi^2 \right) + \frac{\mu^4}{4\lambda}. \tag{5.8}$$

Note that in eq. (5.8) there is a mass term for the H-field, $\mu^2 H^2 \equiv M_H/2H^2$ where we defined[7]

$$M_H = \sqrt{2}\mu \tag{5.9}$$

However, there is *no* mass term for the field ϕ. Thus ϕ is a field for a massless particle called a "Goldstone boson". We will look at this issue in a more general way in section 5.2. Next we consider the kinetic term. We plug eq. (5.7) into $(D_\mu \Phi)^+ D^\mu \Phi$ and get

$$\begin{aligned} (D_\mu \Phi)^+ D^\mu \Phi &= \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} g^2 v^2 A_\mu A^\mu + \frac{1}{2} g^2 A_\mu A^\mu (H^2 + \phi^2) \\ &- g A_\mu \left(\phi \partial_\mu H - H \partial_\mu \phi \right) + g v A_\mu \partial^\mu \phi + g^2 v A_\mu A^\mu H \end{aligned} \tag{5.10}$$

There are several important features in eq. (5.10). Firstly, the gauge boson acquired a mass term $1/2 g^2 v^2 A_\mu A^\mu \equiv 1/2 M_A^2 A_\mu A^\mu$, where we defined

$$M_A = gv \tag{5.11}$$

[7]Note that for a real field ϕ representing a particle of mass m the mass term is $\frac{1}{2}m^2\phi^2$ whereas for a complex field the mass term is $m^2 \phi^\dagger \phi$.

Secondly, there is a coupling of the gauge field to the H-field

$$g^2 v A_\mu A^\mu H = g M_A A_\mu A^\mu H \tag{5.12}$$

It is important to remember that this coupling is proportional to the mass of the gauge boson. Finally, there is also the bilinear term $g\,v\,A^\mu \partial_\mu \phi$, which after integrating by parts (for the action S) may be written as $-\,M_A\,\phi\,\partial_\mu A^\mu$. This simply indicates that the Goldstone boson, ϕ, couples to the longitudinal component of the gauge boson with strength M_A. Let us separate the gauge-boson field A_μ into its transverse and longitudinal components, $A_\mu = A_\mu^L + A_\mu^T$, where $\partial^\mu A_\mu^T = 0$. We can think of the longitudinal component of the gauge boson oscillating between the Goldstone boson with a mixing term given by $-M_A \phi\,\partial^\mu A_\mu^L$, so that the physical particle is described by a superposition of these fields. Later on, we will use the gauge freedom to get rid of this mixing term. There is more than one possibility to do this and we will consider two of them in section 5.3 and 5.4.

5.2 Goldstone Bosons

In the previous subsection we have seen that there is a massless boson, a Goldstone boson, associated with the flat direction in the potential. Goldstones theorem extends this to spontaneous breaking of a general symmetry.

Suppose we have a theory whose Lagrangian is invariant under a symmetry group $\mathcal{G}$ with N generators $\mathbf{T}^a$ and the symmetry group of the vacuum forms a subgroup $\mathcal{H}$ of $\mathcal{G}$, with m generators. This means that the vacuum state is still invariant under transformations generated by the m generators of $\mathcal{H}$, but not the remaining $N-m$ generators of the original symmetry group $\mathcal{G}$. Thus we have

$$\begin{aligned} \mathbf{T}^a|0\rangle &= 0 \qquad a = 1\ldots m \\ \mathbf{T}^a|0\rangle &\neq 0 \qquad a = m+1\ldots N \end{aligned} \tag{5.13}$$

Goldstone's theorem states that then there will be $N-m$ massless particles (one for each broken generator of the group). The case considered in this section is special case in that there is only one generator of the symmetry group (i.e. $N = 1$) which is broken by the vacuum. Thus, there is no generator that leaves the vacuum invariant (i.e. $m = 0$) and we get $N - m = 1$ Goldstone boson.

Like all good general theorems, Goldstone's theorem has a loophole, which arises when one considers a gauge theory, i.e. when one allows the original symmetry transformations to be local. In a spontaneously broken gauge theory, the choice of which vacuum is the true vacuum is equivalent to choosing a gauge, which is necessary in order to be able to quantize the theory. What this means is that the Goldstone bosons, which can, in principle, transform the vacuum into one of the states degenerate with the vacuum, now affect transitions into states which are not consistent with the original gauge choice. This means that the Goldstone bosons are "unphysical" and are often called "Goldstone ghosts".

On the other hand the quantum degrees of freedom associated with the Goldstone bosons are certainly there *ab initio* (before a choice of gauge is made). What happens to them?

A massless vector boson has only two degrees of freedom (the two directions of polarization of a photon), whereas a massive vector (spin-one) particle has three possible values for the helicity of the particle. In a spontaneously broken gauge theory, the Goldstone boson associated with each broken generator provides the third degree of freedom to the gauge bosons. This means that the gauge bosons become massive. The Goldstone boson is said to be "eaten" by the gauge boson. This is exactly what we observed in the previous subsection where we found that there is a mixing term between A_L^μ and ϕ and the physical particle is a superposition of these fields. Thus, in our abelian model, the two degrees of freedom we get by introducing the complex field Φ turn out to be the Higgs field and the longitudinal component of the (now massive) gauge boson. There is no physical, massless particle associated with the degree of freedom ϕ present in Φ.

5.3 The Unitary Gauge

As mentioned above, we want to use the gauge freedom to choose a gauge such that there are no mixing terms between the longitudinal component of the gauge field and the Goldstone boson. To start with, we will use a slightly different parametrization of the complex Φ field in terms of two real fields H and ϕ. Rather than using eq. (5.7) we will use

$$\Phi = \frac{1}{\sqrt{2}}(v+H)\,e^{i\phi/v} = \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi + \dots\right). \tag{5.14}$$

where the dots stand for nonlinear terms in ϕ. Next we make a gauge transformation (see eq. (1.2))

$$\Phi \to \Phi' = e^{-i\phi/v}\,\Phi \tag{5.15}$$

In other words, we fix the gauge such that the imaginary part of Φ vanishes. Under the gauge transformation eq. (5.15) the gauge field transforms according to (see eq. (1.11))

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{gv}[\partial_\mu \phi] \tag{5.16}$$

This is in fact the superposition of A_μ and ϕ that make up the physical field. Note that the change from A_μ to A'_μ made in eq. (5.16) affects only the longitudinal component. If we express now the Lagrangian in terms of Φ' and A'_μ there will be no mixing term. Even better, the ϕ field vanishes altogether! This can easily be seen by noting that under a gauge transformation the covariant derivative $D_\mu\Phi$ transforms in the same way as Φ, thus

$$D_\mu\Phi \to (D_\mu\Phi)' = e^{-i\phi/v} D_\mu\Phi = e^{-i\phi/v}\frac{1}{\sqrt{2}}\left(\partial_\mu H + igA'_\mu(v+H)\right) \tag{5.17}$$

and $(D_\mu\Phi)'^{+}(D^\mu\Phi)'$ is independent of ϕ. Performing the algebra (and dropping the ' for the A-field) we get for the Lagrangian in the unitary gauge

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{M_A^2}{2}A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{M_H^2}{2}H^2 \\ &+ gM_A A_\mu A^\mu H + \frac{g^2}{2}A_\mu A^\mu H^2 - \frac{\lambda}{4}H^4 - \sqrt{\frac{\lambda}{2}}M_H H^3\end{aligned} \tag{5.18}$$

with M_A and M_H as defined in eqs. (5.11) and (5.9) respectively. All the terms quadratic in A_μ may be written (in momentum space) as

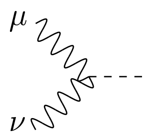
$$A_\mu(-p)\left(-g^{\mu\nu}\,p^2\; + \,p^\mu p^\nu + g^{\mu\nu}M_A^2\right)A_\nu(p). \tag{5.19}$$

The gauge boson propagator is the inverse of the coefficient of $A_\mu(-p)A_\nu(p)$, which is

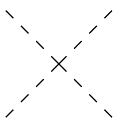
$$-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_A^2}\right)\frac{1}{(p^2 - M_A^2)}, \tag{5.20}$$

which is the usual expression for the propagator of a massive spin-one particle, eq. (4.2). The only other remaining particle is the scalar, H, with mass $m_H = \sqrt{2}\,\mu$ which is the Higgs boson. This is a physical particle, which interacts with the gauge boson and also has cubic and quartic self-interactions. The Lagrangian given in eq. (5.18) leads to the following vertices and Feynman rules.

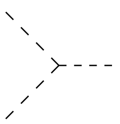
$2\,i\,e^2 g_{\mu\nu}$



$2\,i\,eM_A g_{\mu\nu}$



$6\,i\,\lambda$



$6\,i\,m_H\sqrt{2\lambda}$

The advantage of the unitary gauge is that there appear no unphysical particles, i.e. the ϕ-field has completely disappeared. The disadvantage is that the propagator of the gauge field, eq. (5.20), behaves as p^0 for $p \to \infty$. As discussed in section 4.1 this seems to indicate that the theory is non-renormalizable. It seems that we have not gained anything at all by breaking the theory spontaneously rather than by simply adding a mass term by hand. Fortunately this is not true. In order to see that the theory is still renormalizable, in spite of eq. (5.20), it is very useful to consider a different type of gauges, namely the R_ξ gauges discussed in the next subsection.

5.4 R_ξ Gauges (Feynman Gauge)

The class of R_ξ gauges is a more conventional way to fix the gauge. Recall that in QED we fixed the gauge by adding a term, eq. (1.18) in the Lagrangian. This is exactly what we do here. The gauge fixing term we are adding to the Lagrangian density eq. (5.1) is

$$\begin{aligned}\mathcal{L}_R &\equiv -\frac{1}{2(1-\xi)}\left(\partial_\mu A^\mu - (1-\xi)M_A\phi\right)^2 \\ &= -\frac{1}{2(1-\xi)}\partial_\mu A^\mu \partial_\nu A^\nu + M_A\phi\partial_\mu A^\mu - \frac{1-\xi}{2}M_A^2\phi^2 \end{aligned} \tag{5.21}$$

The special value $\xi = 0$ again corresponds to the Feynman gauge. The second term in eq. (5.21) cancels precisely the mixing term in eq. (5.10). Thus, we achieved our goal. Note however, that in this case, contrary to the unitary gauge, the unphysical ϕ-field does not disappear. The first term in eq. (5.21) is bilinear in the gauge field, thus it contributes to the gauge-boson propagator. The terms bilinear in the A-field are

$$-\frac{1}{2}A^\mu(-p)\left(-g_{\mu\nu}(p^2 - M_a^2) + p_\mu p_\nu - \frac{p_\mu p_\nu}{1-\xi}\right)A^\nu(p) \tag{5.22}$$

which leads to the gauge boson propagator

$$\frac{-i}{(p^2 - M_A^2)}\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2 - (1-\xi)M_A^2}\right) \tag{5.23}$$

Note that in Feynman gauge, the propagator becomes particularly simple. The crucial feature of eq. (5.23), however, is that this propagator behaves as p^{-2} for $p \to \infty$. Thus, this class of gauges is manifestly renormalizable. There is, however, a price to pay. The Goldstone boson is still present. It has acquired a mass, M_A, from the gauge fixing term, and it has interactions with the gauge boson, with the Higgs scalar and with itself. Furthermore, for the purposes of higher order corrections we again need to introduce Faddeev-Popov ghosts which in this case interact not only with the gauge boson, but also with the Higgs scalar and Goldstone boson.

Let us stress that there is no contradiction at all between the apparent non-renormalizability of the theory in the unitary gauge and the manifest renormalizability in the R_ξ gauge. Since physical quantities are gauge invariant, any physical quantity can be calculated in a gauge where renormalizability is manifest. As mentioned above, the price we pay for this is that there are more particles and many more interactions, leading to a plethora of Feynman diagrams. We therefore only work in such gauges if we want to compute higher order corrections. For the rest of these lectures we shall confine ourselves to tree-level calculations and work solely in unitary gauge.

Nevertheless, one cannot over-stress the fact that only when the gauge bosons acquire masses through the Higgs mechanism do we have a renormalizable theory. It is this mechanism that makes it possible to write down a consistent Quantum Field Theory which describes the weak interactions.

5.5 Spontaneous Symmetry Breaking in a Non-Abelian Gauge Theory

It is a relatively straightforward matter to extend this to the case of non-abelian gauge theories. We take as an example an $SU(2)$ gauge theory and consider a complex doublet of scalar fields, $\Phi^i, \ i=1,2$. The Lagrangian density is

$$\mathcal{L} \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + |\mathbf{D}_\mu\Phi|^2 - V(\Phi), \tag{5.24}$$

where [8]

$$\mathbf{D}_\mu\Phi \;=\; \partial_\mu\Phi + i\,g\,W^a_\mu\,\mathbf{T}^a\,\Phi, \tag{5.25}$$

(we have changed notation for the gauge bosons from A^a_μ to W^a_μ), and

$$V(\Phi) \;=\; -\mu^2\Phi^+_i\Phi^i + \lambda\left(\Phi^+_i\Phi^i\right)^2. \tag{5.26}$$

This potential has a minimum at $\Phi^+_i\Phi_i = \frac{1}{2}\mu^2/\lambda$. We choose the vacuum expectation value to be

$$\langle\Phi\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{5.27}$$

($v=\mu/\sqrt{\lambda}$). This vacuum expectation value is not invariant under *any* $SU(2)$ transformation. This means that there is *no* unbroken subgroup, so we expect three Goldstone bosons and all three of the gauge bosons will acquire a mass.

[8] A boldface $\mathbf{D}_\mu$ is a matrix operator with a term proportional to the generators of the group and other terms proportional to the unit matrix. Henceforth we will not explicitly display the unit matrix.

We expand Φ^i about its vacuum expectation value ("vev")

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\,\phi_2 \\ v + H + i\,\phi_3 \end{pmatrix}. \tag{5.28}$$

The ϕ_a, $a = 1 \ldots 3$ are the three Goldstone bosons and H is the physical Higgs scalar. All of these fields have zero vev. If we insert this expansion into the potential eq. (5.26) then we find that we only get a mass term for the Higgs field, with value $m_H = \sqrt{2}\mu$.

For simplicity, we move directly into the unitary gauge by setting all the three ϕ_a to zero. In this gauge $\mathbf{D}_\mu\Phi$ may be written

$$\mathbf{D}_\mu\Phi = \frac{1}{\sqrt{2}} \left(\partial_\mu \begin{pmatrix} 0 \\ H \end{pmatrix} + i\,\frac{g}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^- \\ \sqrt{2}W_\mu^+ & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \right), \tag{5.29}$$

where we have introduced the notation

$$W_\mu^\pm \equiv (W_\mu^1 \pm i\,W_\mu^2)/\sqrt{2} \tag{5.30}$$

and used the explicit form for the generators of $SU(2)$ in the 2×2 representation given by

$$\mathbf{T}^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{T}^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathbf{T}^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.31}$$

The term $|\mathbf{D}_\mu\Phi|^2$ then becomes

$$\begin{aligned} |\mathbf{D}_\mu\Phi|^2 &= \frac{1}{2}\partial_\mu H\,\partial^\mu H + \frac{1}{4}g^2v^2 \left(W_\mu^+ W^{-\,\mu} + \frac{1}{2} W_\mu^3 W^{3\,\mu} \right) \\ &+ \frac{1}{4}g^2 H^2 \left(W_\mu^+ W^{-\,\mu} + \frac{1}{2} W_\mu^3 W^{3\,\mu} \right) + \frac{1}{2}g^2 vH \left(W_\mu^+ W^{-\,\mu} + \frac{1}{2} W_\mu^3 W^{3\,\mu} \right). \end{aligned} \tag{5.32}$$

We see from the terms quadratic in W_μ that all three of the gauge bosons have acquired a mass

$$M_W = \frac{gv}{2} \tag{5.33}$$

5.6 Summary

- In the case of a gauge theory the Goldstone bosons provide the longitudinal component of the gauge bosons, which therefore acquire a mass. The mass is proportional to the magnitude of the vacuum expectation value and the gauge coupling constant. The Goldstone bosons themselves are unphysical.
- It is possible to work in the unitary gauge where the Goldstone boson fields are set to zero.

- When gauge bosons acquire masses by this (Higgs) mechanism, renormalizability is maintained. This can be seen explicitly if one works in a R_ξ gauge, in which the gauge boson propagator decreases as $1/p^2$ as $p \to \infty$. This is a necessary condition for renormalizability. If one does work in such a gauge, however, one needs to work with Goldstone boson fields, even though the Goldstone bosons are unphysical. The number of interactions and the number of Feynman graphs required for the calculation of some process is greatly increased.

6 The Standard Model with one Family

We have now learnt how to write down the Lagrangian of a gauge theory if we know the gauge group and the matter content. In this section we shall apply these ideas and consider the Standard Model with one family.

6.1 Left- and Right- Handed Fermions

The weak interactions are known to violate parity. They are not symmetric under interchange of left-helicity and right-helicity fermions. Thus, in writing down the Standard Model we will have to treat the left-handed and right-handed part separately.

A Dirac field, ψ, representing a fermion, can be expressed as the sum of a left-handed part, ψ_L, and a right-handed part, ψ_R,

$$\psi = \psi_L + \psi_R, \tag{6.1}$$

where

$$\begin{aligned} \psi_L &= P_L \psi \quad \text{with} \quad P_L = \frac{(1-\gamma_5)}{2} &(6.2) \\ \psi_R &= P_R \psi \quad \text{with} \quad P_R = \frac{(1+\gamma_5)}{2} &(6.3) \end{aligned}$$

P_L and P_R are projection operators in the sense that

$$P_L P_L = P_L, \quad P_R P_R = P_R, \quad \text{and } P_L P_R = 0. \tag{6.4}$$

They project out the left-handed (negative) and right-handed (positive) helicity states of the fermion respectively.

The kinetic term of the Dirac Lagrangian and the interaction term of a fermion with a vector field can also be written as a sum of two terms each involving only one helicity.

$$\begin{aligned} \overline{\psi}\gamma^\mu \partial_\mu \psi &= \overline{\psi_L}\gamma^\mu \partial_\mu \psi_L + \overline{\psi_R}\gamma^\mu \partial_\mu \psi_R, &(6.5) \\ \overline{\psi}\gamma^\mu A_\mu \psi &= \overline{\psi_L}\gamma^\mu A_\mu \psi_L + \overline{\psi_R}\gamma^\mu A_\mu \psi_R. &(6.6) \end{aligned}$$

On the other hand, a mass term mixes the two helicities

$$m\overline{\psi}\psi = m\overline{\psi_L}\psi_R + m\overline{\psi_R}\psi_L. \tag{6.7}$$

Exercise 6.1
Use $(\gamma_5)^2 = 1$ to verify eq. (6.4) and $\overline{\psi} = \psi^\dagger\gamma^0$, $\gamma^{5\,\dagger} = \gamma^5$ as well as $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ to verify eq. (6.6).

Thus, if the fermions are massless, we can treat the left-handed and right-handed helicities as separate particles. We can understand this physically from the fact that if a fermion is massive and is moving along the *positive* z-axis along which its spin has a *positive* component, so that the helicity is *positive* in this frame, one can always boost into a frame in which the fermion is moving along the *negative* z-axis, but the component of spin is unchanged. In the new frame the helicity will be *negative*. On the other hand if the particle is massless and travels with the speed of light no such boost is possible and in that case the helicity is a good quantum number.

Exercise 6.2

For a massless spinor

$$u(p) \;=\; \frac{1}{\sqrt{E}}\begin{pmatrix} E\,\chi \\ \underline{\sigma}\cdot\underline{p}\,\chi \end{pmatrix},$$

where χ is a two-component spinor, show that

$$(1 \pm \gamma^5)u(p)$$

are eigenstates of $\underline{\sigma}\cdot\underline{p}/E$ with eigenvalues ± 1 respectively. Take

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and in 4×4 matrix notation $\underline{\sigma}\cdot\underline{p}$ means

$$\begin{pmatrix} \underline{\sigma}\cdot\underline{p} & 0 \\ 0 & \underline{\sigma}\cdot\underline{p} \end{pmatrix}$$

6.2 The Lagrangian

We have made all the preparations to write down a gauge invariant Lagrangian. All we have to do is to pick the gauge group and the matter content of the theory. At this point it has to be stressed that there are no theoretical reasons whatsoever to pick a certain group or certain matter content. It is entirely due to experimental observations that we pick the gauge group in a particular way.

The gauge group for the Standard Model is

$$U(1)_Y \times SU(2) \times SU(3) \tag{6.8}$$

To indicate that the abelian $U(1)$ group is *not* the gauge group of QED but of hypercharge a subscript Y has been added. The corresponding coupling and gauge boson is denoted by g' and B^μ respectively.

The $SU(2)$ group is the isospin group. It has three generators (see eq. (5.31)), the coupling is denoted by g and the three gauge bosons are denoted by $W^1_\mu, W^2_\mu, W^3_\mu$. None of these gauge bosons (and neither B_μ) are physical particles. As we will see, linear combinations of these gauge bosons will make up the photon as well as the W and Z boson.

Finally, the $SU(3)$ is the group of the strong interactions. The corresponding eight gauge bosons are the gluons. In this section we will concentrate on the other two groups. The strong interaction are dealt with in section 3 and the weak interactions of the quarks are deferred to the next section.

After we fixed the gauge group it is time to fix the matter content. For the first family we have

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}; \; u_R; \; d_R; \; \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}; \; e_R; \; \{\nu_R!!\} \tag{6.9}$$

Note that there is a right-handed neutrino ν_R. The Standard Model as it stood until recently did not have a ν_R. However, experimental evidence has become so compelling that we cannot avoid any longer to introduce such a right handed neutrino. Whenever I use the term Standard Model, I mean the 'updated' Standard Model, with matter content as in eq. (6.9). We will often use the notation

$$\ell_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}; \qquad q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}; \tag{6.10}$$

Note that we separated the left- and right-handed components in eq. (6.9). The left-handed components form doublets under $SU(2)$ whereas the right-handed components are singlets. This means that under a $SU(2)$ gauge transformations we have

$$e_R \;\to\; e'_R = e_R \tag{6.11}$$
$$\ell_L \;\to\; \ell'_L = e^{-i\omega^a \mathbf{T}^a} \ell_L \tag{6.12}$$

with the explicit form of $\mathbf{T}^a$ given in eq. (5.31). Thus, the $SU(2)$ singlets e_R, ν_R, u_R and d_R are invariant under $SU(2)$ transformations whereas the doublets ℓ_L and q_L transform in the 'usual' way. The fact that the right-handed particles do not transform under $SU(2)$ means that these particles do not couple to the corresponding gauge bosons $W^1_\mu, W^2_\mu, W^3_\mu$. To see this note that $\overline{e_R}\, \partial\!\!\!/ e_R$ is invariant under the $SU(2)$ gauge transformation. Thus in this particular case the covariant derivative is the same as the usual derivative and does not contain an interaction term.

Since this separation of the electron into its left- and right-handed helicity only makes sense for a massless electron we also need to assume that the electron *is* massless in the exact $SU(2)$ limit and that the mass for the electron arises as a result of spontaneous symmetry breaking in a similar way as the masses for the gauge bosons arise. We will come back to this later.

Under $U(1)_Y$ gauge transformations, the matter fields transform as follows

$$\psi \to \psi' = e^{-i\omega Y(\psi)}\psi \tag{6.13}$$

where Y is the hypercharge of the particle under consideration. The explicit values for the hypercharges of the particles listed in eq. (6.9) are as follows:

$$Y(\ell_L) = -\frac{1}{2};\ Y(e_R) = -1;\ Y(\nu_R) = 0;\ Y(q_L) = \frac{1}{6};\ Y(u_R) = \frac{2}{3};\ Y(d_R) = -\frac{1}{3}; \tag{6.14}$$

Under $SU(3)$ the lepton fields ℓ_L, e_R, ν_R are singlets, i.e. they do not transform at all. This means that they do not couple to the gluons. The quarks on the other hand form triplets under $SU(3)$. The strong interaction does not distinguish between left- and right-handed particles.

We have now listed all particles that belong to the first family, together with their transformation properties under the various gauge transformations. However, since we ultimately want massive weak gauge bosons, we will have to break the $U(1)_Y \times SU(2)$ gauge group spontaneously. Recalling eq. (5.28) we introduce a scalar field, the Higgs field, as a doublet under $SU(2)$:

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ v + H + i\phi_3 \end{pmatrix}. \tag{6.15}$$

The hypercharge is

$$Y(\Phi) = \frac{1}{2} \tag{6.16}$$

and it is a singlet under $SU(3)$. Note that the two complex degrees of freedom in the doublet Φ are split up into 4 real scalar fields ϕ_1, ϕ_2, ϕ_3 and H and that the Higgs field has a vev

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{6.17}$$

with v given in eq. (5.5). The lower entry in eq. (6.15) corresponds exactly to the complex scalar field discussed in section 5.

It is now straightforward to write down a gauge invariant Lagrangian

$$\mathcal{L}_{(SM,1)} = \sum_{\text{gauge bosons}} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{\text{fermions}} \overline{\psi}\, \not{D}\psi + (D_\mu\Phi)^+(D^\mu\Phi) - V(\Phi) \tag{6.18}$$

with the scalar potential as given in eq. (5.4), i.e.

$$V(\Phi) \;=\; -\mu^2\Phi^+\Phi + \lambda \left|\Phi^+\Phi\right|^2 . \tag{6.19}$$

In the first term of eq. (6.18) the sum is over all gauge group generators. In the second term, the sum is over all fermions. Note that the explicit form of the covariant derivative depends on the fermion. For example we have

$$\begin{aligned} D_\mu &= \partial_\mu + ig\mathbf{T}^a W^a_\mu + ig'Y(\ell_L)B_\mu && \text{for } \ell_L && (6.20)\\ D_\mu &= \partial_\mu + ig'Y(e_R)B_\mu && \text{for } e_R && (6.21)\\ D_\mu &= \partial_\mu + ig_s\mathbf{T_s}^a G^a_\mu + ig'Y(d_R)B_\mu && \text{for } d_R && (6.22)\end{aligned}$$

where the strong coupling (g_s), the eight generators of $SU(3)$ ($\mathbf{T_s}^a$) and the corresponding gluon fields (G^a_μ) have been introduced.

We will write down the Lagrangian eq. (6.18) more explicitly and read off the interaction terms in what follows.

Finally, let us stress once more that there are no explicit mass terms for fermions included, since such terms would not be gauge invariant. The fermions will obtain their mass through their interaction with the Higgs doublet. These terms are not yet included in eq. (6.18). Indeed, eq. (6.18) is not the most general gauge invariant Lagrangian that can be written down with the fields at our disposal. We will come back to this point in section 6.5.

6.3 The $(D_\mu\Phi)^+(D^\mu\Phi)$ Term, $W^\pm$, Z and Photons

One of the many terms of the Lagrangian, eq. (6.18), is the kinetic term of the Φ-field, $(D_\mu\Phi)^+(D^\mu\Phi)$. Looking at this term more carefully will help us to understand where the physical gauge bosons, i.e. the $W^\pm$, Z and photons come from and how they are related to the $W^1_\mu, W^2_\mu, W^3_\mu, B_\mu$.

To start with we write the covariant derivative acting on the Higgs doublet, using the notation eq. (5.30). For the moment we will drop all terms involving the ϕ-fields. In fact, in the unitary gauge these degrees of freedom will be eliminated altogether. In the R_ξ gauges, they would lead to interaction terms of the unphysical ϕ-fields with the gauge bosons and the Higgs field.

$$\mathbf{D}_\mu\,\Phi \;=\; \frac{1}{\sqrt{2}}\left(\partial_\mu \;+ i\,\frac{g}{2}\begin{pmatrix} W^3_\mu & \sqrt{2}\,W^-_\mu \\ \sqrt{2}\,W^+_\mu & -W^3_\mu \end{pmatrix} + i\,\frac{g'}{2}B_\mu\right)\begin{pmatrix} 0 \\ v \,+\, H \end{pmatrix}, \tag{6.23}$$

so that

$$|\mathbf{D}_\mu\,\Phi|^2 = \frac{1}{2}(\partial_\mu\,H)^2 + \frac{g^2v^2}{4}\,W^{+\mu}W^-_\mu + \frac{v^2}{8}\left(g\,W^3_\mu - g'B_\mu\right)^2 \;+ \text{interaction terms.} \tag{6.24}$$

where the 'interaction terms' are terms involving three fields (two gauge fields and the H-field). Eq. (6.24) tells us that the W^3_μ and B_μ fields mix (as do W^1_μ and W^2_μ) and the physical gauge bosons must be superpositions of these fields, such that there are no mixing terms. Thus we define

$$Z_\mu \equiv \cos\theta_W\, W^3_\mu - \sin\theta_W B_\mu \tag{6.25}$$

$$A_\mu \equiv \cos\theta_W\, B_\mu + \sin\theta_W W^3_\mu \tag{6.26}$$

with the weak mixing angle θ_W defined by

$$\tan\theta_W \equiv \frac{g'}{g} \tag{6.27}$$

Then eq. (6.24) is rewritten as

$$|\mathbf{D}_\mu\,\Phi|^2 = \frac{1}{2}(\partial_\mu\, H)^2 + \frac{g^2v^2}{2} W^+_\mu W^{-\,\mu} + \frac{v^2g^2}{8\cos^2\theta_W} + 0 A_\mu A^\mu \tag{6.28}$$

Here we see how the $SU(2)$ and $U(1)$ are unified (or at least 'entangled') in the sense that the neutral gauge boson that acquires a mass through the Higgs mechanism is the linear superposition of a gauge boson from the $SU(2)$ and the $U(1)_Y$ gauge boson.

From eq. (6.28) we can read off the masses of the gauge bosons. The last term tells us that the linear combination eq. (6.26) remains massless. This field is identified with the photon. For the other fields we have

$$M_W = \frac{1}{2} gv; \qquad M_Z = \frac{1}{2}\frac{gv}{\cos\theta_W} \tag{6.29}$$

The Z boson mediates the neutral current weak interactions These were not observed until after the development of the model. From the magnitude of amplitudes involving weak neutral currents (exchange of a Z boson), one can infer the magnitude of the weak mixing angle, θ_W. The ratio of the mass of the Z and W bosons is a prediction of the Standard Model. More precisely, we define a quantity known as the ρ-parameter by

$$M_W^2 = \rho\, M_Z^2 \cos^2\theta_W. \tag{6.30}$$

In the Standard Model $\rho = 1$ in leading order of perturbation theory. In higher orders there is some small correction to this and the actual Standard Model prediction is $\rho = 1.01$. Note that the ρ-parameter would be very different from one if the symmetry breaking were due to a scalar multiplet which was not a doublet of weak isospin.

Exercise 6.3

Using the experimental values, $M_W = 80.39 \pm 0.07$ Gev, $M_Z = 91.187 \pm 0.002$ GeV, and $\sin^2\theta_W = 0.2316 \pm 0.0002$, calculate the experimental value of ρ (together with its experimental error).

The spontaneous symmetry breaking mechanism breaks $SU(2) \times U(1)_Y$ down to $U(1)$. It is this surviving $U(1)$ that is identified as the $U(1)$ of electromagnetism. It is not the $U(1)_Y$ of the original gauge group but a set of transformations generated by a particular linear combination of the original $U(1)$ and rotations about the third axis of weak isospin. To see this we note that the explicit representation of the generator $\mathbf{Y}$ as a 2×2 matrix that can be combined with the explicit representation of $\mathbf{T}^1$, $\mathbf{T}^2$ and $\mathbf{T}^3$, eq. (5.31), is given by

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.31}$$

The factor 1/2 ensures the normalization condition eq. (2.2). Using eqs. (5.31) and (6.31) it can easily be seen that the symmetry associated with the generator

$$Q \equiv Y + T^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.32}$$

is not broken, i.e. $Q|0\rangle = 0$ (see eq. (4.5)). Thus, starting with four generators, we get only three Goldstone bosons. These will become the longitudinal component of three gauge bosons, thereby giving them a mass, but the fourth is left massless.

The coupling of any particle to the photon is always proportional to

$$g \sin\theta_W \, (Y \ + \ T_3) \ = \ g \sin\theta_W \, Q \tag{6.33}$$

Thus we can identify $g \sin\theta_W$ with one unit of electric charge and we have a relationship between the weak coupling, g, and the electron charge, e,

$$e \ = \ g \sin\theta_W \tag{6.34}$$

.

6.4 Weak Interactions of Leptons

If we now look at the fermionic terms (an isodoublet of left-handed fermions with hypercharge $Y = -\frac{1}{2}$, l_L and a right-handed electron with weak hypercharge $Y = -1$), the fermionic part of the Lagrangian is

$$\mathcal{L}_{\text{Fermi}} = i\,\overline{\ell_L}^T \gamma^\mu \mathbf{D}_\mu \,\ell_L + i\overline{e_R}\,\gamma^\mu D_\mu \,e_R + i\overline{\nu_R}\,\gamma^\mu \partial_\mu \,\nu_R \tag{6.35}$$

where the covariant derivatives in the two terms are replaced by eqs. (6.20) and (6.21) respectively. This gives the following interaction terms between the leptons and the gauge bosons:

$$-\frac{g}{2}\begin{pmatrix}\overline{\nu_L}\\ \overline{e_L}\end{pmatrix}^T\gamma^\mu\left(\begin{pmatrix}W^0_\mu & \sqrt{2}\,W^-_\mu\\ \sqrt{2}\,W^+_\mu & -W^0_\mu\end{pmatrix}-\tan\theta_W B_\mu\right)\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\;-\;i\,g\,\tan\theta_W\,\overline{e_R}\,\gamma^\mu B_\mu e_R. \tag{6.36}$$

Expanding this out and using eqs. (6.25) and (6.26) to replace the B_μ and W^3_μ by the physical particles Z_μ and A_μ and writing out the projection operators for left- and right-handed fermions, eqs. (6.2) and (6.3), we obtain the following interactions:

1. A coupling of the charged vector bosons $W^\pm$, which mediate transitions between neutrinos and electrons with an interaction term

$$-\frac{g}{2\sqrt{2}}\,\overline{\nu}\,\gamma^\mu\left(1-\gamma^5\right)e\,W^-_\mu\;+\text{h.c.} \tag{6.37}$$

   (h.c. means 'hermitian conjugate' and gives the interaction involving an emitted W^+_μ where the incoming particle is a neutrino and the outgoing particle is an electron.

2. The usual coupling of the electron with the photon (using the relation eq. (6.34))

$$g\,\sin\theta_W\,\overline{e}\,\gamma^\mu\,e\,A_\mu. \tag{6.38}$$

   Note that the left- and right-handed electrons have exactly the same coupling to the photon so that the electromagnetic coupling turns out to be purely vector (i.e. no γ^5 term).

3. The coupling of neutrinos to the neutral weak gauge boson, Z_μ

$$-\frac{g}{4\cos\theta_W}\,\overline{\nu}\,\gamma^\mu\left(1-\gamma^5\right)\,\nu\,Z_\mu \tag{6.39}$$

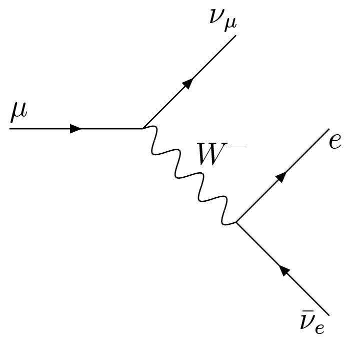
4. The coupling of both the left- and right-handed electron to the Z

$$\frac{g}{4\cos\theta_W}\,\overline{e}\,\left(\gamma^\mu\left(1-\gamma^5\right)-4\sin^2\theta_W\gamma^\mu\right)\,e\,Z_\mu \tag{6.40}$$

We can include other lepton families, the muon and its neutrino, and the tau-lepton with its neutrino, simply as copies of what we have for the electron and its neutrino[9]. For each family we have a weak isodoublet of left-handed leptons and a right-handed isosinglet for the charged lepton.

Thus, the mechanism which determines the decay of the muon (μ) is one in which the muon converts into its neutrino and emits a charged W^-, which then decays into an electron and (anti)neutrino. The Feynman diagram is

[9] We are neglecting mixing for the moment



The amplitude for this process is given by the product of the vertex rules for the emission (or absorption) of a W^- with a propagator for the W boson between them. Up to corrections of order m_μ^2/M_W^2, we may neglect the effect of the term $q^\mu q^\nu/M_W^2$ in the W-boson propagator, so that we have

$$\left(-i\frac{g}{2\sqrt{2}}\overline{\nu_\mu}\,\gamma^\rho(1-\gamma^5)\mu\right)\left(\frac{-\,i\,g_{\rho\sigma}}{q^2-M_W^2}\right)\left(-\,i\,\frac{g}{2\sqrt{2}}\,\overline{e}\,\gamma^\sigma(1-\gamma^5)\,\nu_e\right), \tag{6.41}$$

where q is the momentum transferred from the muon to its neutrino. Since this is negligible in comparison with M_W we may neglect it and the expression for the amplitude simplifies to

$$i\,\frac{g^2}{8M_W^2}\;\overline{\nu_\mu}\,\gamma^\rho(1-\gamma^5)\mu\;\overline{e}\,\gamma_\rho(1-\gamma^5)\nu_e. \tag{6.42}$$

Before the development of this model, weak interactions were described by the "four-fermi model" with a weak interaction Hamiltonian given by

$$\mathcal{H}_{ijkl} \;=\; \frac{G_F}{\sqrt{2}}\;\overline{\psi}_i\,\gamma^\mu(1-\gamma^5)\psi_j\;\overline{\psi}_k\,\gamma_\mu(1-\gamma^5)\psi_l. \tag{6.43}$$

We now recognize this as an effective low-energy Hamiltonian which may be used when the energy scales involved in the weak process are negligible compared with the mass of the W boson. The Fermi coupling constant, G_F is related to the electric charge, e, the W mass and the weak mixing angle by

$$G_F \;=\; \frac{e^2}{4\sqrt{2}\,M_W^2\,\sin^2\theta_W}. \tag{6.44}$$

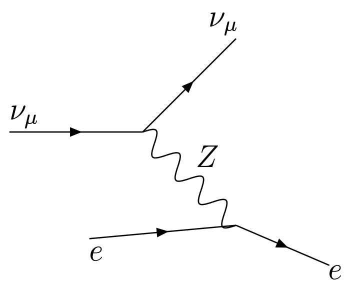
This gives us a value for G_F,

$$G_F \;= 1.12\times 10^{-5}\ \mathrm{GeV}^{-2}, \tag{6.45}$$

which is very close to the value of 1.17×10^{-5} GeV^{-2}, measured from the lifetime of the muon.

We see that the weak interactions are 'weak', not because the coupling is particularly small (the $SU(2)$ gauge coupling is about twice as large as the electromagnetic coupling), but because the exchanged boson is very massive, so that the Fermi coupling constant of the four-fermi theory is very small. The large mass of the W boson is also responsible for the fact that the weak interactions are short range (of order 10^{-18} m).

In the Standard Model, however, we also have neutral weak currents. Thus, for example, we can have elastic scattering of muon-type neutrinos against electrons via the exchange of the Z boson. The Feynman diagram for such a process is



Exercise 6.4
Let us write the four-fermi interaction for this process as

$$\mathcal{H} \;=\; \frac{G_F}{\sqrt{2}}\,\overline{\nu_e}\,\gamma^\rho(1-\gamma^5)\nu_e\,\overline{\mu}\,\gamma_\rho(v\;-\;a\gamma^5)\mu,$$

where v and a give us the vector and axial-vector coupling of the muon to the Z-boson (the muon couples in an identical way to the electron). Determine v and a in terms of θ_W.

6.5 Fermion Masses and Yukawa Couplings

We cannot have an explicit mass term for the electrons since a mass term mixes left-handed and right-handed fermions and we have assigned these to different multiplets of weak isospin.

We can, however, have an interaction between the left-handed lepton doublet, the right-handed electron (or muon or tau-lepton) and the scalar doublet Φ. This is one of the gauge invariant terms that have been left out in eq. (6.18). Such an interaction is called a 'Yukawa interaction' and is written

$$\mathcal{L}_{\text{Yukawa}} \;=\; -\,G_e\,\overline{l_L}^{i}\,\Phi_i e_R \;+\; \text{h.c.} \tag{6.46}$$

Note that this term has zero weak hypercharge. In the unitary gauge this is

$$\mathcal{L}_{\text{Yukawa}} \;=-\,\frac{G_e}{\sqrt{2}}\begin{pmatrix}\overline{\nu_L}\\ \overline{e_L}\end{pmatrix}^T\begin{pmatrix}0\\ v\;+\;H\end{pmatrix}e_R\;+\;\text{h.c.} \tag{6.47}$$

The part proportional to the vev is simply

$$-\frac{G_e\,v}{\sqrt{2}}\left(\overline{e_L}\,e_R+\overline{e_R}\,e_L\right) \;=\; \frac{G_e\,v}{\sqrt{2}}\,\overline{e}\,e, \tag{6.48}$$

and we see that the electron has acquired a mass, which is proportional to the vev of the scalar field. This immediately gives us a relation for the Yukawa coupling in terms of the electron mass, m_e, and the W mass, M_W,

$$G_e \;=\; g\,\frac{m_e}{\sqrt{2}\,M_W}. \tag{6.49}$$

There is, moreover, a Yukawa coupling between the electron and the scalar Higgs field

$$-\,g\,\frac{m_e}{2\,M_W}\;\overline{e}\,H\,e \tag{6.50}$$

This coupling is proportional to the mass of the electron. Thus, as for the gauge bosons, the strength of the coupling of the Higgs to fermions is proportional to the mass of the fermions.

Note that for the moment there is *no* coupling between the neutrino and the Higgs, and no neutrino mass term. This is the Standard Model as it stood until recently. However, as mentioned previously, we have to update the Standard Model in order to account for neutrino masses and mixing. We will discuss this in more detail in section 8.

6.6 Classifying the Free Parameters

The free parameters in the Standard Model for one generation of leptons are

- The two gauge couplings for the $SU(2)$ and $U(1)$ gauge groups, g and g'.
- The two parameters, μ and λ in the scalar potential $V(\Phi)$
- The Yukawa coupling constant, G_e.

It is convenient to replace these five parameters by five parameters which are directly measurable experimentally, namely e, $\sin\theta_W$, and three masses m_H, M_W, m_e. The relation between these physical parameters and the parameters of the initial Lagrangian are

$$\tan\theta_W \;=\; \frac{g'}{g}, \tag{6.51}$$

$$e \;=\; g\sin\theta_W, \tag{6.52}$$

$$m_H \;=\; \sqrt{2}\mu, \tag{6.53}$$

$$M_W \;=\; \frac{g\,\mu}{2\sqrt{\lambda}}, \tag{6.54}$$

$$m_e \;=\; G_e\frac{\mu}{\sqrt{\lambda}}. \tag{6.55}$$

Note that when we add more generations of leptons we acquire one more parameter for each added generation, namely the Yukawa coupling, which determines the mass of the charged lepton.

In terms of these measured quantities, the Z mass, M_Z, and the Fermi-coupling, G_F are *predictions* (although historically G_F was known for many years before the discovery of the W boson and its value was used to predict the W mass).

6.7 Summary

- Weak interactions are mediated by the gauge bosons of weak isospin which acts only on the left-handed helicity components of fermions.

- The neutrino and left-handed component of the electron form a doublet of this weak isospin whereas the right-handed component of the electron is a singlet.

- There is also a weak hypercharge $U(1)_Y$ gauge symmetry. Both left- and right-handed leptons transform under this $U(1)_Y$ with a hypercharge which is related to the electric charge of the electron by the relation eq. (6.34).

- The scalar multiplet that is responsible for the spontaneous symmetry breaking also carries weak hypercharge. As a result, neutral gauge bosons mix in such a way that the superposition eq. (6.25) acquires a mass whereas its orthogonal superposition eq. (6.26) remains massless. The massive particle is the Z boson and the massless particle is the photon. The magnitude of the electron charge, e, is then given by $e = g \sin\theta_W$.

- The weak interactions proceed via the exchange of massive charged or neutral gauge bosons. The old four-fermi weak Hamiltonian is an effective Hamiltonian which is valid for low energy processes in which all momenta are small compared with the W mass. The Fermi coupling is obtained in terms of e, M_W and $\sin\theta_W$ by the relation eq. (6.44).

- In the symmetry limit the leptons are massless. The spontaneous symmetry breaking mechanism that gives a vev to the scalar field also generates a mass for the electron.

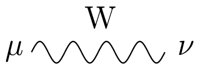
- A second generation (μ, ν_μ) and third generation (τ, ν_τ) of leptons are added to the model as copies. The only difference is that the Yukawa coupling is different for each generation in order to generate the appropriate masses for the muon and tau-lepton respectively.

For completeness, a full set of Feynman rules for the case of a single family of leptons is given in the appendix to this lecture.

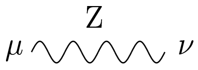
Feynman Rules in the Unitary Gauge (for one Generation of Leptons)

Propagators:

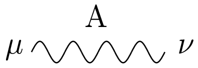
All propagators carry momentum p.



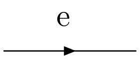
$-i\,(g_{\mu\nu} - p_\mu\, p_\nu/M_W^2)/(p^2 - M_W^2)$



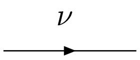
$-i\,(g_{\mu\nu} - p_\mu\, p_\nu/M_Z^2)/(p^2 - M_Z^2)$



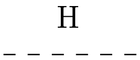
$-i\,g_{\mu\nu}/p^2$



$i\,(\gamma\cdot p + m_e)/(p^2 - m_e^2)$



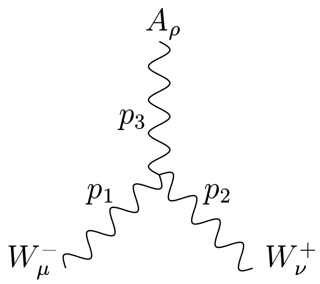
$i\,\gamma\cdot p/p^2$



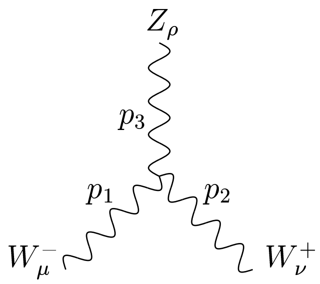
$i/(p^2 - m_H^2)$

Three-point gauge-boson couplings:

All momenta are incoming

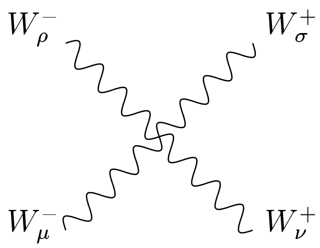


$$i\, g\, \sin\theta_W\, \left((p_1 - p_2)_\rho\, g_{\mu\nu} + (p_2 - p_3)_\mu\, g_{\nu\rho} + (p_3 - p_1)_\nu\, g_{\rho\mu}\right)$$

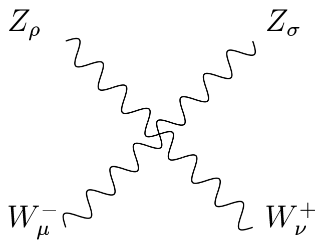


$$i\, g\, \cos\theta_W\, \left((p_1 - p_2)_\rho\, g_{\mu\nu} + (p_2 - p_3)_\mu\, g_{\nu\rho} + (p_3 - p_1)_\nu\, g_{\rho\mu}\right)$$

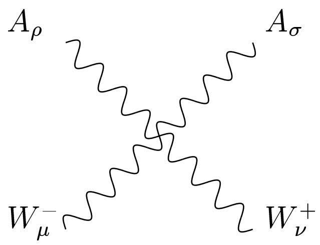
Four-point gauge-boson couplings:



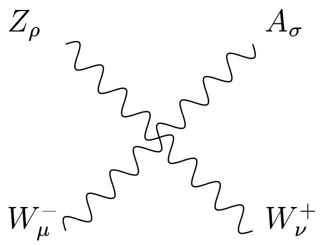
$$i\,g^2\;(2g_{\mu\rho}\,g_{\nu\sigma}\;-\;g_{\mu\nu}\,g_{\rho\sigma}\;-\;g_{\mu\sigma}\,g_{\nu\rho})$$



$$i\,g^2\cos^2\theta_W\;(2g_{\mu\nu}\,g_{\rho\sigma}\;-\;g_{\mu\rho}\,g_{\nu\sigma}\;-\;g_{\mu\sigma}\,g_{\nu\rho})$$

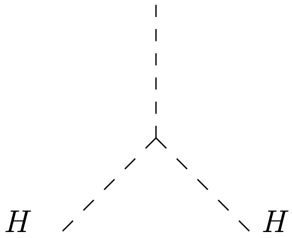


$$i\,g^2\sin^2\theta_W\;(2g_{\mu\nu}\,g_{\rho\sigma}\;-\;g_{\mu\rho}\,g_{\nu\sigma}\;-\;g_{\mu\sigma}\,g_{\nu\rho})$$

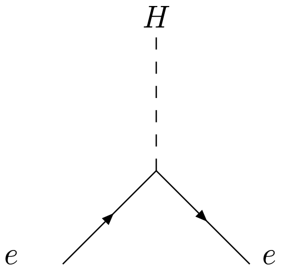


$$i\,g^2\cos\theta_W\;\sin\theta_W\;(2g_{\mu\nu}\,g_{\rho\sigma}\;-\;g_{\mu\rho}\,g_{\nu\sigma}\;-\;g_{\mu\sigma}\,g_{\nu\rho})$$

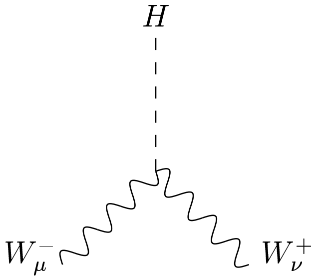
Three-point couplings with Higgs scalars:



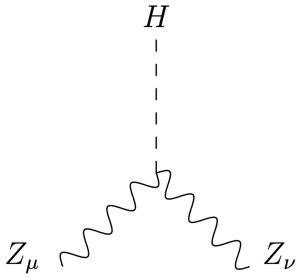
$-\frac{3}{2}\, i\, g\, m_H^2/M_W$



$-\frac{1}{2}\, i\, g\, m_e/M_W$

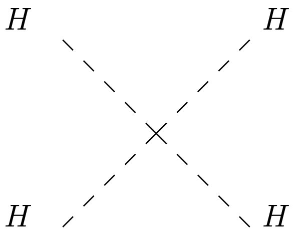


$i\, g\, M_W\, g_{\mu\nu}$

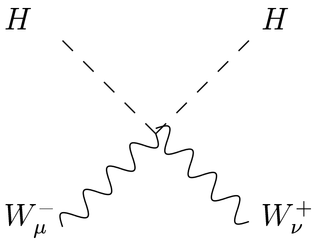


$i\ (g/\cos^2\theta_W)\ M_W\, g_{\mu\nu}$

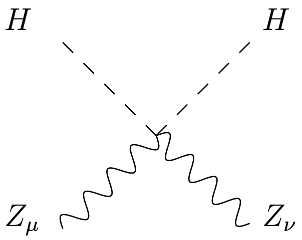
Four-point couplings with Higgs scalars:



$-\frac{3}{4}\, i\, g\, (m_H^2/M_W^2)$

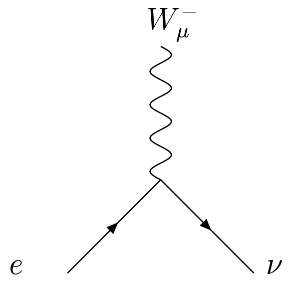


$\frac{1}{2}\, i\, g^2\, g_{\mu\nu}$

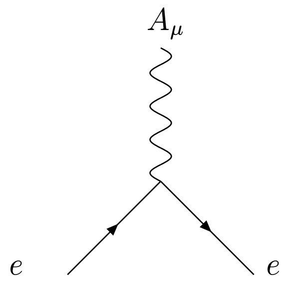


$\frac{1}{2}\, i\ (g^2/\cos^2\theta_W)\ g_{\mu\nu}$

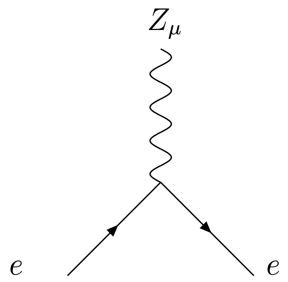
Fermion interactions with gauge bosons:



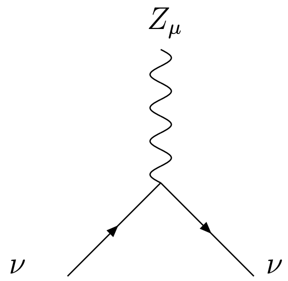
$$-i\,\left(g/2\sqrt{2}\right)\,\gamma_\mu\,(1-\gamma^5)$$



$$+i\,g\,\sin\theta_W\,\gamma_\mu$$



$$+\tfrac{1}{4}\,i\,(g/\cos\theta_W)\,\gamma_\mu\,\left(1-4\,\sin^2\theta_W\,-\gamma^5\right)$$



$$-\tfrac{1}{4}\,i\,(g/\cos\theta_W)\,\gamma_\mu\,(1-\gamma^5)$$

7 Electroweak Interactions of Hadrons

In the previous section we have given the (incomplete) Lagrangian of the Standard Model with one family and given the interactions of the leptons with the gauge bosons and the Higgs. We will now focus on the interactions of the quarks, first for one family only and later extending to three families.

7.1 One Generation of Quarks

We incorporate hadrons into the Standard Model by adding weak isodoublets of left-handed quarks and weak isosinglets of right-handed quarks, as given in eq. (6.9). A single generation consists of a u-quark and a d-quark. Remember that each of these is a triplet of color $SU(3)$. We suppress this color index for the moment but we must bear in mind that as far as the electroweak interactions are concerned we are really adding three copies of each of these quarks.

Thus we have an isodoublet

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$$

and two isosinglets, u_R and d_R.

We recall that the hypercharges of the quarks are as follows: The isodoublet, q_L has $Y = \frac{1}{6}$, whereas u_R and d_R have $Y = +\frac{2}{3}$ and $Y = -\frac{1}{3}$ respectively (there is *no* reason why the right-handed components of the u and d quarks should have the same weak hypercharge, although the left-handed components *must* have the same weak hypercharge since they transform into each other under weak isospin transformations). According to eq. (6.32) this results in electric charges of $+\frac{2}{3}$ and $-\frac{1}{3}$ for the u and d quarks respectively.

The covariant derivatives are therefore given by:

$$\mathbf{D}_\mu q_L = \left(\partial_\mu + i\, g\, W_\mu^a \mathbf{T}^a + i\frac{1}{6} g \tan\theta_W\ B_\mu\right) q_L, \tag{7.1}$$

$$D_\mu u_R = \left(\partial_\mu + i\frac{2}{3} g \tan\theta_W\ B_\mu\right) u_R, \tag{7.2}$$

$$D_\mu d_R = \left(\partial_\mu - i\frac{1}{3} g \tan\theta_W\ B_\mu\right) d_R. \tag{7.3}$$

We insert the explicit expressions eqs. (7.1), (7.2) and (7.3) into the $\overline{\psi}\ \not{D}\psi$ term of the Lagrangian eq. (6.18). Expressing the gauge fields in terms of the physical gauge fields

$$\begin{aligned} W_\mu^3 &= \cos\theta_W\, Z_\mu + \sin\theta_W\, A_\mu \\ B_\mu &= \cos\theta_W\, A_\mu - \sin\theta_W\, Z_\mu \end{aligned} \tag{7.4}$$

(see eqs. (6.25) and (6.26)) leads to the following interaction terms of the quarks with the gauge bosons:

1. The coupling of the charged vector bosons $W^{\pm}$, which mediate transitions between u and d quarks are analogous to the lepton case. The interaction term is

$$-\frac{g}{2\sqrt{2}}\,\overline{u}\,\gamma^{\mu}\left(1-\gamma^{5}\right)d\,W_{\mu}^{-}\;+\text{h.c.} \tag{7.5}$$

2. The couplings of the quarks with the photon give us the required quark charges,

$$-\frac{2}{3}\,g\,\sin\theta_W\,\overline{u}\,\gamma^{\mu}\,u\,A_{\mu}\;+\;\frac{1}{3}\,g\,\sin\theta_W\,\overline{d}\,\gamma^{\mu}\,d\,A_{\mu}. \tag{7.6}$$

3. The coupling of the quarks to the Z can be written in the general form

$$-\frac{g}{2\cos\theta_W}\,\overline{q_i}\,\left(T_i^3\,\gamma^{\mu}\left(1-\gamma^{5}\right)-\,2Q_i\sin^2\theta_W\gamma^{\mu}\right)\,q_i\,Z_{\mu}, \tag{7.7}$$

   where quark i has third component of weak isospin T_i^3 and electric charge Q_i.

From these terms in the Lagrangian we can read off directly the Feynman rules for the three-point vertices with two quarks and one weak gauge boson. Then we can use these vertices to calculate weak interactions at the quark level. This allows us, for example, to calculate the total decay width of the Z or W boson, by calculating the decay width into all possible quarks and leptons. However, for inclusive processes, in which we trigger on known initial or final state hadrons, information is needed about the probability to find a quark with given properties inside an initial hadron or the probability that a quark with given properties will decay ("fragment") into a final state hadron.

Exercise 7.1

The decay rate for the Z into a fermion-antifermion pair, $Z \to f\,\bar{f}$, is

$$\Gamma \;=\; \frac{1}{2M_Z}\int d^{LIPS}\,|\mathcal{M}|^2 \;=\; \frac{1}{64\,\pi^2\,M_Z}\int d\Omega\,|\mathcal{M}|^2,$$

where d^{LIPS} means integrate over the phase-space for the two final-state fermions, and $d\Omega$ means integrate over the solid angle of one final-state particle. Write the general interaction term for the coupling of the Z boson to a fermion as

$$-\frac{g}{2\cos\theta_W}\gamma^\mu\left(v_f - a_f\,\gamma^5\right).$$

Show that the square matrix element, summed over the fermion spins and averaged over the boson spins is

$$|\mathcal{M}|^2 \;=\; -\frac{1}{12}\,g_{\mu\nu}\,\frac{g^2}{\cos^2\theta_W}\,\left((v_f)^2 + (a_f)^2\right)\,\mathrm{Tr}\left(\gamma^\mu\gamma\cdot k_1\gamma^\nu\gamma\cdot k_2\right),$$

where k_1 and k_2 are the momenta of the outgoing fermions and the gauge polarization sum is

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}\,\epsilon_\nu^{(\lambda)} \;=\; -g_{\mu\nu} + \frac{q_\mu\,q_\nu}{M_Z^2}$$

($q = k_1 + k_2$ is the initial momentum of the Z boson). Hence show that

$$\Gamma \;=\; \frac{1}{48\,\pi}\frac{g^2}{\cos^2\theta_W}\,\left((v_f)^2 + (a_f)^2\right)\,M_Z.$$

Neglect the masses of the fermions in comparison with the Z mass.

Exercise 7.2

The Z boson can decay leptonically into a pair of neutrinos or charged leptons of all three generations and hadronically into u quarks, d quarks, c quarks, s quarks, or b quarks (c quarks couple like u quarks, whereas s quarks and b quarks couple like d quarks). Deduce the values of v_f and a_f for each of these cases and consequently estimate the decay width of the Z boson. (The experimental value is 2.49 GeV.)

[Take M_Z=91 GeV, $\sin^2\theta_W$=0.23, and the fine-structure constant, $\alpha = 1/129$ (why this value ?)].

7.2 Quark Masses

The quarks are also assumed to be massless in the symmetry limit and acquire a mass through the spontaneous symmetry breaking mechanism, via their Yukawa coupling with the scalars. The interaction term

$$- G_d \, \overline{q_L}^{\,i} \, \Phi_i \, d_R \; + \; \text{h.c.} \tag{7.8}$$

gives a mass, m_d to the d quark, when we replace Φ_i by its vev. This mass is given by

$$m_d \;=\; \frac{G_d}{\sqrt{2}} v \;=\; \sqrt{2}\, \frac{G_d \, M_W}{g} \tag{7.9}$$

However, this does *not* generate a mass for the u quark.

In the case of $SU(2)$ there is a second way in which we can construct an invariant from such a Yukawa interaction. This is through the term

$$- G_u \, \epsilon_{ij} \, \overline{q_L}^{\,i} \, \Phi^{\dagger j} u_R + \; \text{h.c.} \qquad (i, j \;=\; 1, 2), \tag{7.10}$$

where ϵ_{ij} is the two-dimensional antisymmetric tensor. Note that this term also has zero weak hypercharge as required by the $U(1)$ symmetry. This term does indeed give a mass m_u to the u quark, where

$$m_u \;=\; \frac{G_u}{\sqrt{2}} v \;=\; \sqrt{2}\, \frac{G_u \, M_W}{g} \tag{7.11}$$

The Higgs scalar couples to both the u quark and the d quark, with interaction terms

$$- g \frac{m_u}{2\, M_W} \, \overline{u} \, H \, u \; - \; g \frac{m_d}{2\, M_W} \, \overline{d} \, H \, d. \tag{7.12}$$

7.3 Adding Another Generation

The second generation of quarks consists of a c quark, which has electric charge $+\frac{2}{3}$ and an s quark, with electric charge $-\frac{1}{3}$. We can just add a copy of the left-handed isodoublet and copies of the right-handed singlets in order to include this generation.

The only difference would be in the Yukawa interaction terms where the coupling constants are chosen to reproduce the correct masses for the new quarks. But in this case there is a further complication. It is possible to write down Yukawa terms which mix quarks of different generations, e.g.

$$- G_{ds} \left(\begin{array}{c} \overline{u_L} \\ \overline{d_L} \end{array} \right)^T \Phi \, s_R \tag{7.13}$$

This term gives rise to a mass mixing between the d quark and s quark.

The physical particles are those that diagonalize the mass matrix. It is therefore more convenient to work in a basis where the mass matrix does *not* contain off-diagonal terms and the quarks involved in the Yukawa terms are the physical quarks such that there is *no* Yukawa mixing between quarks of different generations.

This means that the quarks which couple to the gauge bosons *are*, in general, superposition of physical quarks. Thus we write the two isodoublets of left-handed quarks as

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L , \tag{7.14}$$

and

$$\begin{pmatrix} c \\ \tilde{s} \end{pmatrix}_L , \tag{7.15}$$

where $\tilde{d}$ and $\tilde{s}$ are related to the physical d quark and s quark by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \end{pmatrix} = \mathbf{V_C} \begin{pmatrix} d \\ s \end{pmatrix}, \tag{7.16}$$

where $\mathbf{V_C}$ is a unitary 2×2 matrix.

Terms which are diagonal in the quarks are unaffected by this unitary transformation of the quarks. Thus the coupling to photons or Z bosons is the same whether written in terms of $\tilde{d}, \ \tilde{s}$ or simply $s, \ d$. We return to this later.

On the other hand the coupling to the charged gauge bosons is

$$-\frac{g}{2\sqrt{2}} \overline{u} \gamma^\mu (1-\gamma^5) \tilde{d} W_\mu^- \ - \frac{g}{2\sqrt{2}} \overline{c} \gamma^\mu (1-\gamma^5) \tilde{s} W_\mu^- \quad + \text{h.c.} \tag{7.17}$$

which we may write as

$$-\frac{g}{2\sqrt{2}} \begin{pmatrix} \overline{u} \\ \overline{c} \end{pmatrix}^T \gamma^\mu (1-\gamma^5) \mathbf{V_C} \begin{pmatrix} d \\ s \end{pmatrix} W_\mu^- \quad + \text{h.c.} \tag{7.18}$$

We see now that there is no point in applying the transformation to *both* the $T^3 = +\frac{1}{2}$ and $T^3 = -\frac{1}{2}$ quarks, since we can absorb the product of two unitary matrices into what we call $\mathbf{V_C}$ here.

The most general 2×2 unitary matrix may be written as

$$\begin{pmatrix} e^{-i\gamma} & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} e^{i\alpha} & \\ & e^{i\beta} \end{pmatrix}. \tag{7.19}$$

We have set one of the phases to unity since we can always absorb an overall phase by adjusting the remaining phases, α, β, and γ.

The phases, α, β, γ can be absorbed by performing a global phase transformation on the d, s and u quarks respectively. This again has no effect on the neutral terms. Thus the most general observable unitary matrix is given by

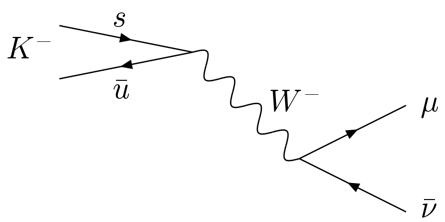
$$\mathbf{V_C} \;=\; \left(\begin{array}{cc} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{array} \right), \tag{7.20}$$

where θ_C is the Cabibbo angle.

In terms of the physical quarks, the charged gauge boson interaction terms are

$$\begin{aligned} -\frac{g}{2\sqrt{2}} \Big(& \cos\theta_C\, \overline{u}\,\gamma^\mu(1-\gamma^5)\, d \;+\; \sin\theta_C\, \overline{u}\,\gamma^\mu(1-\gamma^5)\, s \\ & +\; \cos\theta_C\, \overline{c}\,\gamma^\mu(1-\gamma^5)\, s \;-\; \sin\theta_C\, \overline{c}\,\gamma^\mu(1-\gamma^5)\, d \Big)\, W_\mu^- \quad + \text{h.c.} \end{aligned} \tag{7.21}$$

This means that the u quark can undergo weak interactions in which it is converted into an s quark, with an amplitude that is proportional to $\sin\theta_C$. It is this that gives rise to strangeness violating weak interaction processes, such as the leptonic decay of K^- into a muon and antineutrino. The Feynman diagram for this process is



7.4 Flavor Changing Neutral Currents

Although there are charged weak interactions that violate strangeness conservation, there are no known neutral weak interactions that violate strangeness. For example, the K^0 does not decay into a muon pair or two neutrinos (branching ratio $<\ 10^{-5}$). This means that the the Z boson only interacts with quarks of the same flavor. We can see this by noting that the Z-boson interaction terms are unaffected by a unitary transformation. This absence of flavor changing neutral currents (FCNC) in experimental data is rather important. As we will see, in the Standard Model there are no FCNC at tree level and the absence of FCNC is an important constraint for any extension of the Standard Model.

The Z-boson interactions with d and s quarks are proportional to

$$\overline{\tilde{d}}\,\tilde{d} + \overline{\tilde{s}}\,\tilde{s}. \tag{7.22}$$

(we have suppressed the γ-matrices which act between the fermion fields). Writing this out in terms of the physical quarks we get

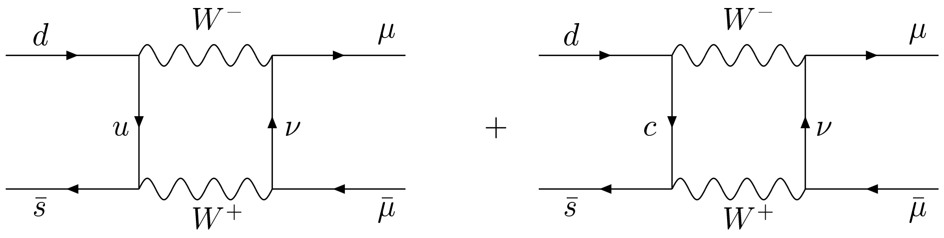
$$\cos^2\theta_C\,\overline{d}\,d \;+\; \sin\theta_C\,\cos\theta_C\,\overline{s}\,d \;+\; \cos\theta_C\,\sin\theta_C\,\overline{d}\,s \;+\; \sin^2\theta_C\,\overline{s}\,s$$
$$+\;\cos^2\theta_C\,\overline{s}\,s \;-\; \sin\theta_C\,\cos\theta_C\,\overline{d}\,s \;-\; \cos\theta_C\,\sin\theta_C\,\overline{s}\,d \;+\; \sin^2\theta_C\,\overline{d}\,d. \qquad (7.23)$$

We see that the cross terms cancel out and we are left with simply

$$\overline{d}\,d \;+\; \overline{s}\,s. \qquad (7.24)$$

This cancellation is the reason for the absence of FCNC and is simply a consequence of the unitarity of the mixing matrix eq. (7.19). This effect is also known as the "GIM" (Glashow-Iliopoulous-Maiani) mechanism. It was used to predict the existence of the c quark.

There can be a small contribution to strangeness changing neutral processes from higher order corrections in which we do not exchange a Z boson, but two charged W bosons. The Feynman diagrams for such a contribution to the leptonic decay of a K^0 (which consists of a d quark and an s antiquark) are



These diagrams differ in the flavor of the internal quark which is exchanged, being a u quark in the first diagram and a c quark in the second. Both of these diagrams are allowed because of the Cabibbo mixing. The first of these diagrams gives a contribution proportional to

$$+\sin\theta_C\,\cos\theta_C,$$

which arises from the product of the two couplings involving the emission of the W bosons. The second diagram gives a term proportional to

$$-\cos\theta_C\,\sin\theta_C.$$

If the c quark and u quark had identical mass then these two contributions would cancel precisely. However, because the c quark is much more massive than the u quark, there is some residual contribution. This was used to limit the mass of the c quark to < 5 GeV, before it was discovered.

7.5 Adding a Third Generation

Adding a third generation is achieved in the same way. In this case the three weak isodoublets of left-handed fermions are

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}, \quad \begin{pmatrix} c \\ \tilde{s} \end{pmatrix}, \quad \begin{pmatrix} t \\ \tilde{b} \end{pmatrix}, \tag{7.25}$$

where $\tilde{d}$, $\tilde{s}$ and $\tilde{b}$ are related to the physical d, s and b quarks by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \\ \tilde{b} \end{pmatrix} = \mathbf{V_{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{7.26}$$

The 3×3 unitary matrix $\mathbf{V_{CKM}}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Once again it only affects the charged weak processes in which a W boson is exchanged. For this reason the elements are written as

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{7.27}$$

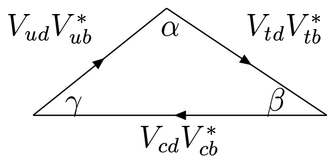
A 3×3 unitary matrix can have nine independent parameters (counting the real and imaginary parts of a complex element as two parameters). In this case there are six possible fermions involved in the charged weak processes and so we can have five relative phase transformations, thereby absorbing five of the nine parameters.

This means that whereas the Cabibbo matrix only has one parameter (the Cabibbo angle, θ_C) the CKM matrix has four independent parameters. If the CKM matrix were real it would only have three independent parameters. This means that in the case of the CKM matrix some of the elements may be complex. The four independent parameters can be thought of as three mixing angles between the three pairs of generations and a complex phase.

The requirement of unitarity puts various constraints on the elements of the CKM matrix. For example we have

$$V_{ud}\, V_{ub}^* + V_{cd}\, V_{cb}^* + V_{td}\, V_{tb}^* = 0$$

This can be represented as a triangle in the complex plane known as the "unitarity triangle"



The angles of the triangle are related to ratios of elements of the CKM matrix

$$\alpha = -\arg\left\{\frac{V_{td}\,V_{tb}^*}{V_{ud}\,V_{ub}^*}\right\} \tag{7.28}$$

$$\beta = -\arg\left\{\frac{V_{td}\,V_{tb}^*}{V_{cd}\,V_{cb}^*}\right\} \tag{7.29}$$

$$\gamma = -\arg\left\{\frac{V_{ud}\,V_{ub}^*}{V_{cd}\,V_{cb}^*}\right\} \tag{7.30}$$

A popular representation of the CKM matrix is due to Wolfenstein and uses parameters A, which is assumed to be of order unity, a complex number, $(\rho + i\,\eta)$ and a small number λ, which is approximately equal to $\sin\theta_C$. In terms of these parameters the CKM matrix is written

$$\mathbf{V_{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\,\lambda^3\,(\rho - i\,\eta) \\ -\lambda & 1-\lambda^2/2 & A\,\lambda^2 \\ A\lambda^3\,(1-\rho-i\,\eta) & -A\,\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{7.31}$$

We see that whereas the W bosons can mediate a transition between a u quark and a b quark (V_{ub}) or between a t quark and a d quark (V_{td}), the amplitude for such transitions are suppressed as the cube of the small quantity which determines the amplitude for transitions between the first and second generations, λ. The $\mathcal{O}(\lambda^4)$ corrections are needed to ensure the unitarity of the CKM matrix and these corrections have several matrix elements which are complex.

7.6 CP Violation

The possibility that some of the elements of the CKM matrix may be complex provides a mechanism for the violation of CP conservation. Violation of CP conservation has been observed in the K^0, $\overline{K}^0$ system, and is currently being investigated for B-mesons.

Higher-order corrections to the masses of B^0 and $\overline{B^0}$, give rise to mixing between the two states. Thus the mass matrix can be written

$$\begin{pmatrix} M_{B^0} & \Delta M \\ (\Delta M)^* & M_{B^0} \end{pmatrix}. \tag{7.32}$$

The mass eigenstates are therefore

$$|B_L\rangle = p\,|B^0\rangle + q\,|\overline{B}^0\rangle, \tag{7.33}$$

whose mass is $M - \frac{1}{2}\Delta m$ and

$$|B_H\rangle = p\,|B^0\rangle - q\,|\overline{B}^0\rangle, \tag{7.34}$$

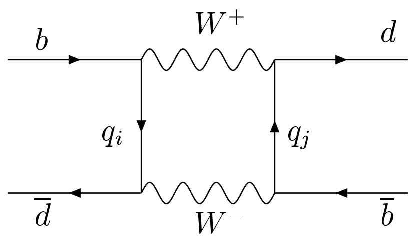
whose mass is $M + \frac{1}{2}\Delta m$.

If ΔM were real then we would have $p = q = 1/\sqrt{2}$ and these mass eigenstates would be CP eigenstates, using the fact that

$$CP|B^0\rangle \;=\; -|\overline{B}^0\rangle.$$

However, the non-zero phases in the CKM matrix give rise to a complex phase for ΔM, so that the ratio of p and q is a complex phase, indicating that B_L and B_H are *not* CP eigenstates.

A typical weak interaction contribution to the mass-mixing term, ΔM is given by the Feynman diagram



Note that on the left we have a B^0, consisting of a b quark and an d antiquark, whereas on the right we have a $\overline{B}^0$ consisting of an d quark and a b antiquark. The internal quarks marked q_i and q_j can each be u, c or t quarks and each of the vertices carries some element of the CKM matrix. The total contribution, therefore, may be written

$$\sum_{i=u,c,t}\;\sum_{j=u,c,t} V_{ib}\,V_{id}^*\,V_{jb}^*\,V_{jd}\,a_{ij}\,.$$

Once again, if all the masses of the quarks were equal then the amplitudes a_{ij} would all be equal, and this would vanish by the unitarity constraints imposed on the elements V_{ik}. Since the quarks do not all have the same mass, there is some residual contribution. Indeed, the above diagram is dominated by the term in which a t quark is exchanged on both sides, since this quark is much more massive than the rest.

Restricting ourselves to the t quark exchange contribution, we can read off the phase of this contribution, without calculating the diagram itself. It is given by the phase of the products, of the CKM matrix elements entering in the diagram, namely

$$(V_{td}^*\,V_{tb})^2.$$

The phase of this quantity is the square of the ratio of p and q, so we have

$$\frac{p}{q} \;=\; \frac{V_{td}^*\,V_{tb}}{V_{td}\,V_{tb}^*}$$

Now suppose that at time $t = 0$ we prepare a state which is purely B^0. Accounting for the fact that the B^0 meson has a decay rate Γ, we can use eqs.(7.33, 7.34) to write the state at time t as

$$|B(t)\rangle \ = \ e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) |B^0\rangle \ + \ i\frac{q}{p} \sin\left(\frac{\Delta m}{2} t\right) |\overline{B}^0\rangle . \right) \qquad (7.35)$$

Now suppose that the amplitude for a state $|B^0\rangle$ to decay into some CP eigenstate $|f\rangle$ is A_f, whereas the amplitude for a state $|\overline{B}^0\rangle$ to decay into the state $|f\rangle$ is $\overline{A}_f$. Once again, if CP were conserved, we would have

$$A_f \ = \ \pm \overline{A}_f,$$

but the CP violating phases give rise to a more general complex phase for the ratio of these two amplitudes.

This means that the amplitude to find the state $|f\rangle$ after time t is given by

$$\langle f | \mathcal{H}_{wk} | B(t) \rangle \ = \ e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) A_f \ + \ i\frac{q}{p} \sin\left(\frac{\Delta m}{2} t\right) \overline{A}_f \right). \qquad (7.36)$$

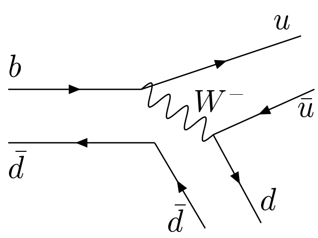
Similarly, if we had prepared a $\overline{B}^0$ at $t = 0$ the amplitude to find the state $|f\rangle$ would be

$$\langle f | \mathcal{H}_{wk} | \overline{B}(t) \rangle \ = \ e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) \overline{A}_f \ - \ i\frac{p}{q} \sin\left(\frac{\Delta m}{2} t\right) A_f \right). \qquad (7.37)$$

Taking the square moduli, to find the decay rates we arrive at the result

$$\frac{\Gamma(B(t) \to f) - \Gamma(\overline{B}(t) \to f)}{\Gamma(B(t) \to f) + \Gamma(\overline{B}(t) \to f)} \ = \ - \ \sin(\Delta m\, t)\, \Im m \left(\frac{q}{p} \frac{\overline{A}_f}{A_f} \right). \qquad (7.38)$$

For example, if the state $|f\rangle$ is the CP even two-pion state $|\pi^0 \pi^0\rangle$, the Feynman diagram at the quark level for $A_{2\pi}$ is



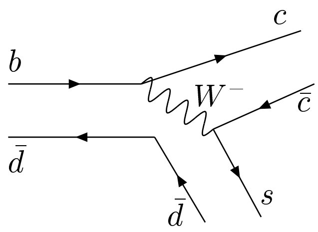
To calculate the decay amplitudes we would need to know the wave functions for the mesons in terms of the constituent quark-antiquark pairs, but for the ratio $\overline{A}_{2\pi}/A_{2\pi}$ we just need the ratios of the CKM matrix elements occurring in this diagram, namely

$$\frac{\overline{A}_{2\pi}}{A_{2\pi}} \ = \ \frac{V_{ub}\, V_{ud}^*}{V_{ub}^*\, V_{ud}},$$

so that (using eq. (7.28))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{2\pi}}{A_{2\pi}}\right) = \frac{V_{td}\,V_{tb}^*\,V_{ub}\,V_{ud}^*}{V_{td}^*\,V_{tb}\,V_{ub}^*\,V_{ud}} = -\sin(2\alpha) \tag{7.39}$$

As a further example, we consider the so-called "golden channel" where $|f\rangle$ is the state $|J/\psi\, K_S\rangle$. In this case the quark level Feynman diagram is



Here there is a further complication since the outgoing state $(s\,\bar{d})$ is actually $\overline{K}^0$ (and likewise for the $\overline{B}^0$ decay it would be a K^0). As in the B^0 system, the mass eigenstates are given by

$$\begin{aligned} |K_S\rangle &= p_K|K^0\rangle + q_K|\overline{K}^0\rangle \\ |K_L\rangle &= p_K|K^0\rangle - q_K|\overline{K}^0\rangle. \end{aligned} \tag{7.40}$$

Once again, if CP were conserved we would have $p_K = q_K = 1/\sqrt{2}$, and these mass eigenstates would be eigenstates of CP. The phases in the CKM matrix introduce a phase in the ratio of p_K and q_K, calculated from diagrams similar to the ones for the B^0 system (but with the b quark replaced by an s quark). In this case it is the diagram with internal c quark exchange that dominates (although the mass of the c quark is much less than the t quark the CKM matrix elements are much larger for c quark exchange than t quark exchange and this effect dominates), so we have a factor

$$\frac{q_K}{p_K} = \frac{V_{cd}^*\,V_{cs}}{V_{cd}\,V_{cs}^*}$$

which enters in the ratio of the decay amplitudes, giving

$$\frac{\overline{A}_{J/\psi\,K_S}}{A_{J/\psi\,K_S}} = -\frac{V_{cb}\,V_{cs}^*}{V_{cb}^*\,V_{cs}}\frac{V_{cd}^*\,V_{cs}}{V_{cd}\,V_{cs}^*} = -\frac{V_{cb}\,V_{cd}^*}{V_{cb}^*\,V_{cd}},$$

(a minus occurs because the $J/\psi\,K_S$ state is CP odd) so that (using eq. (7.29))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{J/\psi\,K_S}}{A_{J/\psi\,K_S}}\right) = -\frac{V_{td}\,V_{tb}^*\,V_{cb}\,V_{cd}^*}{V_{td}^*\,V_{tb}\,V_{cb}^*\,V_{cd}} = \sin(2\beta)\,. \tag{7.41}$$

7.7 Summary

- A left-handed quark doublet has weak hypercharge $Y = \frac{1}{6}$, and the right-handed components have weak hypercharges equal to the electric charges of the quarks themselves.

- In order to generate a mass for the $T^3 = +\frac{1}{2}$ quark, one has to add a different type of Yukawa coupling given by eq. (7.10).

- A second generation may be added as a copy, but in this case one can have mass-mixing between quarks of different generations. In terms of the mass eigenstates, the charged W bosons mediate transitions between a $T^3 = +\frac{1}{2}$ quark (u or c) and a superposition of $T^3 = -\frac{1}{2}$ quarks (d and s). This mechanism allows weak interactions that violate strangeness conservation. The mixing matrix has only one independent parameter, the Cabibbo angle.

- The unitarity of the mixing matrix guarantees that there are no strangeness changing neutral processes. Weak interactions involving the exchange of a Z boson do not change flavor. There is a small violation of this in higher orders owing to the mass splitting between the quarks.

- Including a third generation, the mixing matrix for the $T^3 = -1/2$ quarks (d, s and b) is the CKM matrix. This matrix has has four independent parameters, so that some of the matrix elements may be complex.

- The possibility that some of the elements of the CKM matrix may be complex leads to a weak interaction contribution to a mass mixing of B^0 and $\overline{B}^0$ which can be complex. This gives rise to CP violation, since the eigenstates of the B^0 mass matrix are then no longer eigenstates of CP. The CKM matrix also introduces phases in the ratios of the decay amplitudes for B^0 and $\overline{B}^0$ to a given CP eigenstate. Products of the phase of the mass mixing and the ratio of the decay amplitudes can be observed directly in tagged B-meson experiments and the angles α and β of the unitarity triangle can be directly measured.

8 Neutrinos

In this section we will be dealing with neutrinos, in particular their masses and oscillations. Until recently, both these topics would have been 'beyond the Standard Model' topics, but by now experimental evidence is so strong that we have to update the Standard Model and include right-handed neutrinos. This is no problem at all. In fact we have already included them in eq. (6.9) and we will happily keep calling the new version of the model Standard Model. However, now it is time to consider what effect the introduction of ν_R has.

8.1 Neutrino Oscillation

Neutrino oscillations are very closely connected to neutrino masses. In fact, if we include a ν_R nothing prevents us from writing a mass term, eq. (6.7), for neutrinos. Once we have a mass term we have to carefully distinguish between mass eigenstates and interaction eigenstates (see eqs. (7.16) and (7.26)). If the mass eigenstates are not the same as the interaction eigenstates we can have mixing.

For simplicity we will consider the case of two flavors only which we will take to be the electron and muon flavor. Of course, in the Standard Model we have three families, but the important concepts can all be understood in the simpler case. In sections 7.3 and 7.5 we denoted the mass eigenstates of quarks by normal letters q and the interaction eigenstates by $\tilde{q}$. Here we will sometimes use a different notation. The mass eigenstates Ψ will be denoted by $|1\rangle$ and $|2\rangle$ respectively, whereas the interaction eigenstates $\tilde{\Psi}$ will be denoted by $|\alpha\rangle$ and $|\beta\rangle$ respectively. They are related by an equation very similar to eq. (7.16)

$$\begin{pmatrix} |\alpha\rangle \\ |\beta\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}; \qquad \text{or} \quad \tilde{\Psi} = V \cdot \Psi \tag{8.1}$$

In the mass-eigenbasis we have the Schrödinger equation

$$i\frac{d}{dt}\Psi = H \cdot \Psi \tag{8.2}$$

with a diagonal Hamiltonian

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \tag{8.3}$$

This Schrödinger equation can easily be solved. Defining our initial states at $t = 0$ as $|1\rangle \equiv |1(t=0)\rangle, |2\rangle \equiv |2(t=0)\rangle$ we get the time dependent states

$$\begin{aligned} |1(t)\rangle &= e^{-iE_1 t}|1\rangle \\ |2(t)\rangle &= e^{-iE_1 t}|2\rangle \end{aligned} \tag{8.4}$$

Let us repeat the last few steps in the interaction eigenbasis. Multiplying eq. (8.2) by V from the left we get the corresponding Schrödinger equation as

$$i\frac{d}{dt}\tilde{\Psi} = \tilde{H} \cdot \tilde{\Psi} \tag{8.5}$$

with

$$\tilde{H} \equiv V \cdot H \cdot V^{-1} = \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \tag{8.6}$$

where

$$\begin{aligned} a &= \frac{1}{2}(E_1 + E_2) &(8.7)\\ b &= \frac{1}{2}(E_1 - E_2)\cos(2\theta) &(8.8)\\ c &= -\frac{1}{2}(E_1 - E_2)\sin(2\theta) &(8.9) \end{aligned}$$

The crucial feature of the new Hamiltonian is that it is no longer diagonal. As a result, if we start at time $t = 0$ with an interaction eigenstate $|\alpha\rangle$, at a later time we get a superposition of $|\alpha\rangle$ and $|\beta\rangle$ interaction eigenstates. Indeed, using eq. (8.1) for the time dependent states we get

$$\begin{aligned} |\alpha(t)\rangle &= e^{-iE_1 t}\cos\theta\,|1\rangle + e^{-iE_2 t}\sin\theta\,|2\rangle &(8.10)\\ |\beta(t)\rangle &= -e^{-iE_1 t}\sin\theta\,|1\rangle + e^{-iE_2 t}\cos\theta\,|2\rangle &(8.11) \end{aligned}$$

Let us now use this relation to compute the oscillation probability $\mathcal{P}_{\alpha\to\beta}(t)$. What we mean by this is the following: assume that at $t = 0$ we know that our state is a pure interaction eigenstate $|\alpha\rangle$. To be concrete we can assume this is an electron neutrino ν_e created in the sun. $\mathcal{P}_{\alpha\to\beta}(t)$ then gives us the probability that at a later time t this sate has evolved into an interaction eigenstate $|\beta\rangle$. Of course, this probability is simply the absolute value squared of the amplitude

$$\begin{aligned} \mathcal{P}_{\alpha\to\beta}(t) &= |\langle\beta|\alpha(t)\rangle|^2 \\ &= \left|-\sin\theta\,\cos\theta\left(e^{-iE_1 t} - e^{-iE_2 t}\right)\right|^2 \\ &= \frac{1}{2}\sin^2(2\theta)\,(1 - \cos(E_2 - E_1)t) \\ &= \sin^2(2\theta)\sin^2\left(\frac{E_2 - E_1}{2}\,t\right) &(8.12) \end{aligned}$$

In the first step we used eq. (8.10) and the orthogonality of the mass eigenstates $\langle i|j\rangle = \delta_{ij}$. The expression for $\mathcal{P}_{\alpha\to\beta}(t)$ can be brought into a more useful form by noting that

$$E_i = \sqrt{p^2 + m_i^2} = p + \frac{m_i^2}{2p} + \ldots \tag{8.13}$$

and, therefore,

$$\frac{1}{2}(E_2 - E_1) = \frac{m_2^2 - m_1^2}{4E} \equiv \frac{\Delta m^2}{4E} \tag{8.14}$$

where E is the energy of the beam[10]. Furthermore, since the neutrinos travel at the speed of light, we have $L = vt = ct = t$, where L is the distance travelled by the neutrino. Thus, we arrive at the final expression for the oscillation probability

$$\mathcal{P}_{\alpha\to\beta}(t) = \sin^2(2\theta)\sin^2\left(L\,\frac{\Delta m^2}{4E}\right) \tag{8.15}$$

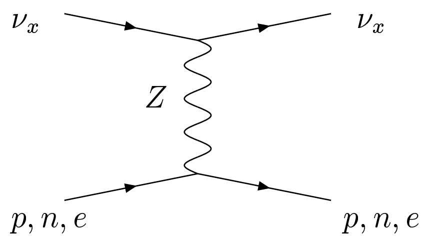
Eq. (8.15) has the expected properties in that the probability vanishes for $L \to 0$, $\theta \to 0$ and most notably for $\Delta m^2 \to 0$. This last limit tells us that there is no mixing if the two neutrino species have the same mass and, in particular, if they are massless.

8.2 The MSW or Matter Effect

So far we have considered oscillations in vacuum, i.e. we assumed that the neutrinos were travelling through the vacuum. While this is true for most of the time, the neutrinos produced in the sun first have to travel through the sun before they can reach us. The matter surrounding the neutrinos can have a crucial effect on the oscillation probability for the neutrinos. This effect is called the matter effect or the Mikheyev-Smirnov-Wolfenstein effect.

The question at the heart of the problem is: how does the Hamiltonian $\tilde{H}$, eq. (8.6), change through interactions of the neutrinos with surrounding matter? There are basically neutral and charged current interactions.

As we have learnt, neutral current interactions are mediated by the exchange of a Z boson. Taking into account that the surrounding matter is basically made of protons, neutrons and electrons a typical Feynman diagram is

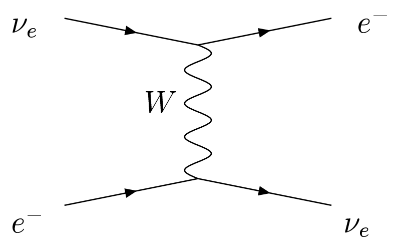


The important point is that these interactions are independent on the flavor x of the neutrino. Thus they affect the two diagonal entries of the Hamiltonian in the same way. This means

[10] This argument can be made more rigorously using wave packets

they change a, eq. (8.7), i.e. the Hamiltonian is modified by $a \to \bar{a}$. As we will see later, this change is irrelevant.

The charged current interactions are mediated by a $W^{\pm}$. A typical Feynman diagram is



These interactions take place only for electron neutrinos since there are no μ (or τ) in the surrounding matter. In our convention where we identify the $|\alpha\rangle$ state with an electron neutrino, this means that only the top-left entry of the Hamiltonian, eq. (8.6) is modified. Thus, including the matter effects we arrive at the following Hamiltonian

$$\tilde{H}_{rmMSW} = \begin{pmatrix} \bar{a}+b+w & c \\ c & \bar{a}-b \end{pmatrix} \tag{8.16}$$

where w comes from the charged current interactions. The explicit form of w is not important for us. What we want to know is how the w-term modifies the mixing angle. To find the modified mixing angle $\theta_{\rm MSW}$ we have to diagonalize $\tilde{H}_{\rm MSW}$, i.e. we have to find

$$V_{\rm MSW} = \begin{pmatrix} \cos\theta_{\rm MSW} & \sin\theta_{\rm MSW} \\ -\sin\theta_{\rm MSW} & \cos\theta_{\rm MSW} \end{pmatrix} \tag{8.17}$$

such that

$$H_{\rm MSW} \equiv V_{\rm MSW}^{-1} \cdot \tilde{H}_{\rm MSW} \cdot V_{\rm MSW} \tag{8.18}$$

is diagonal. If we plug the explicit forms for $V_{\rm MSW}$, eq. (8.17), and $\tilde{H}_{\rm MSW}$, eq. (8.16), into eq. (8.18) we find for the off-diagonal terms

$$c\cos(2\theta_{\rm MSW}) + \frac{2b+w}{2}\sin(2\theta_{\rm MSW}) \tag{8.19}$$

which vanishes for

$$\tan(2\theta_{\rm MSW}) = -\frac{2c}{2b+w} = \frac{-\Delta m^2 \sin(2\theta)}{4Ew - \Delta m^2\cos(2\theta)} \tag{8.20}$$

where we have used eqs. (8.8) and (8.9).

We note that $\theta_{\rm MSW}$ does not depend on a, thus as mentioned above, the change $a \to \bar{a}$ induced by the neutral current interactions does not matter at all. The important point is that for $4Ew \sim \Delta m^2 \cos(2\theta)$ there can be a dramatic effect and the oscillation probability can increase substantially. In fact, this effect is very important in the explanation of experimental results.

8.3 The See-Saw Mechanism

In this section we are concerned with neutrino masses and offer a possible explanation why they might be so small compared to other fermion masses. We will restrict ourselves to the case of one family.

As mentioned previously, introducing a right-handed neutrino allows us to write down the same kind of Yukawa coupling as for the u-type quarks, eq. (7.10). This will result in a 'usual' Dirac mass term for the neutrinos of the form

$$m_D\,\overline{\nu}\nu = m_D(\overline{\nu}_L\nu_R + \overline{\nu}_R\nu_L) \tag{8.21}$$

(compare to eq. (7.11)). There is no doubt that such a term has to be introduced in the Lagrangian, but it leads immediately to the question why the ν mass is so much smaller than the other fermion masses. In fact, we would expect that the Yukawa couplings of all fermions are roughly of the same order. This would lead to neutrino masses roughly of the same size as the masses of the other leptons, obviously in sharp contrast to observations.

However, the very special properties of the right-handed neutrinos allow us to write down yet another term in the Lagrangian. Recall that we have to write down the most general gauge invariant Lagrangian, given the gauge group and the matter content. In fact, since the ν_R is a singlet under all gauge transformations, we can (have to!!) add the term

$$M\nu_R\nu_R + \text{h.c.} \tag{8.22}$$

Note that for this term to be gauge invariant it is mandatory that $Y(\nu_R) = 0$ and that ν_R neither couples to $SU(2)$ nor $SU(3)$ gauge bosons. Such a right-handed neutrino is said to be sterile.

The term eq. (8.22) also leads to a mass term, called a Majorana mass. Whereas m_D is expected to be of the same size as charged-lepton masses the most natural value for M is much larger. Ultimately we expect that at a high energy scale (maybe the GUT scale $M \sim 10^{15}$ GeV) there is a theory that explains all the fermion masses. Then, the natural value for the fermion masses is of the order M. However, all fermion masses except for the

ν_R are protected by the chiral symmetry. This explains why $m_D \ll M$. To understand the consequences of $M \gg m_D$ consider the neutrino mass matrix for (ν_L, ν_R).

$$\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix} \tag{8.23}$$

In order to get the masses of the physical particles, i.e. the eigenstates of the mass matrix, we have to diagonalize this matrix. The eigenvalues are approximately given by

$$\frac{m_D^2}{M} \quad \text{and} \quad M \tag{8.24}$$

where we used $m_D \ll M$. Thus we can see that the physical neutrinos are a (nearly) left-handed neutrino with mass m_D^2/M and a (nearly) right-handed neutrino with mass M. Since $m_D \sim m_{\text{leptons}}$ and $M \gg m_D$ this provides us with an explanation why the mass of the left-handed neutrino is so much smaller than the mass of the other leptons. If this explanation is correct, there should also be a very heavy (nearly) sterile right-handed neutrino.

Finally we note that if we allow for three families there will be mixing between the various families as in the quark case. In the quark case we had the CKM-mixing matrix. In the lepton sector, the matrix is denoted by V_{MNS}, after Maki, Nakagawa and Sakata. Whereas in the quark sector we had four parameters, resulting in one complex phase, in the neutrino case we have three complex phases. This difference is due to the Majorana nature of the neutrinos. Thus, there are additional sources for CP-violation in the Standard Model.

8.4 Summary

- If a right-handed neutrino is introduced, we have to distinguish between mass eigenstates and interaction eigenstates of neutrinos in the same way as we did for d-type quarks.
- If the different neutrino species have different mass they can oscillate between various interaction eigenstates. Thus, an electron neutrino produced in the sun can arrive as a tau neutrino on earth.
- If the neutrinos travel through matter rather than the vacuum the oscillation pattern can change dramatically.
- The see-saw mechanism provides us with an explanation why the neutrino masses are so much smaller than the other lepton masses.

9 Supersymmetry

This is the only section truly beyond the Standard Model. However, since supersymmetry (susy) plays such an important role in particle physics, it deserves some space, even in a Standard Model course. In this section we will outline the basic ideas of this new symmetry, why so many theorists like it and we will illustrate the ideas in the simplest supersymmetric model, namely the susy harmonic oscillator.

9.1 Extending the Poincaré Symmetry

As we have seen, symmetries play a crucial role in particle physics. Gauge invariance and Lorentz invariance are the guiding principles that lead us to the Standard Model. Therefore, it is natural to ask, if it is possible to extend these symmetries.

In order to formulate this question more precisely we have to spend some time on recalling what we mean by a symmetry. Let's start with the gauge symmetries. We pick a certain group with some generators $\mathbf{T}^a$. These generators satisfy a certain algebra, eq. (2.4),

$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c \tag{9.1}$$

The corresponding transformations are given by $e^{-i\omega^a \mathbf{T}^a}$ where ω^a are the parameters of the transformation. We say the theory has a gauge symmetry, if its Lagrangian is invariant under these transformations[11].

The situation concerning invariance under Lorentz transformations or more general under Poincaré transformations is very similar. The group of the Poincaré transformations is made up of translations and Lorentz transformations. There are four generators, $\mathbf{P}_\mu$, for the translations (one for each space-time direction) and six generators for Lorentz transformations $\mathbf{M}_{\mu\nu}$ (rotations around the three axes and boosts along the three axes). The generators $\mathbf{M}_{\mu\nu} = -\mathbf{M}_{\nu\mu}$ are antisymmetric, thus even though μ and ν run from 0 to 3, there are only six such generators. As for the generators of the gauge group, these generators of the Poincaré group satisfy a certain algebra.

$$\begin{aligned}
[\mathbf{P}_\mu, \mathbf{P}_\nu] &= 0 && (9.2)\\
[\mathbf{M}_{\mu\nu}, \mathbf{M}_{\rho\sigma}] &= -i\left(g_{\mu\rho}\mathbf{M}_{\nu\sigma} + g_{\nu\sigma}\mathbf{M}_{\mu\rho} - g_{\mu\sigma}\mathbf{M}_{\nu\rho} - g_{\nu\rho}\mathbf{M}_{\mu\sigma}\right) && (9.3)\\
[\mathbf{P}_\mu, \mathbf{M}_{\nu\rho}] &= i\left(g_{\mu\nu}\mathbf{P}_\rho - g_{\mu\rho}\mathbf{P}_\nu\right) && (9.4)
\end{aligned}$$

[11] For the sake of completeness we should mention that the various fields in the Lagrangian can transform under different representations of the group

The detailed form of this algebra is of no importance to us. All we have to know is that the $\mathbf{M}$ and $\mathbf{P}$ generators mix, that is to say they have non-trivial (i.e. right hand side not equal to 0) commutation relations. The finite transformations generated by $\mathbf{M}$ and $\mathbf{P}$ are $e^{-i\omega^{\mu\nu}\mathbf{M}_{\mu\nu}}$ and $e^{-ix^{\mu}\mathbf{P}_{\mu}}$ with six parameters $\omega_{\mu\nu}$ (three rotation angles and three boost parameters) and four parameters x^{μ}. The crucial difference is that the gauge parameters ω^{a} are allowed to depend on x whereas the 10 parameters of the Poincaré transformations are global[12].

The all important question now is whether we can enlarge the symmetry given in eq. (9.4), i.e. whether we could add more generators and require invariance under the symmetry transformations generated by the larger group. We can answer this immediately in the positive, since the gauge symmetries, eq. (9.1) do precisely this. However, such an extension is called a 'trivial' extension. The term trivial relates to the fact that all additional generators $\mathbf{T}^{a}$ have trivial commutation relations with the Poincaré generators, i.e.

$$\begin{aligned} [\mathbf{M}_{\mu\nu}, \mathbf{T}^{a}] &= 0 \\ [\mathbf{P}_{\mu}, \mathbf{T}^{a}] &= 0 \end{aligned} \tag{9.5}$$

Let me stress that trivial does not mean useless or not interesting. In fact, gauge symmetries are of utmost importance. But what we're after here is a true, non-trivial extensions with additional generators, such that at least some of them have non-trivial commutation relations with the 10 Poincaré generators.

There is an important no-go theorem, the Coleman-Mandula theorem which tells us that under very general conditions a non-trivial extension of the Poincaré group is not possible. At this point we would have to stop, were it not for the fact that all these no-go theorems have always a back door. In fact, the theorem relies on using bosonic generators only. A bosonic generator is a generator which acting on a bosonic (fermionic) state always produces another bosonic (fermionic) state, i.e. it does not alter the spin of the state it acts on.

If we do not restrict ourselves to bosonic generators then the Poincaré symmetry can be enlarged. Roughly speaking, there is only one way to do this (Haag-Lopuszanski-Sohnius theorem) namely to make the theory supersymmetric. What this means is that we introduce fermionic generators $\mathbf{Q}_{\alpha}$ satisfying the following algebra

$$\{\mathbf{Q}_{\alpha}, \overline{\mathbf{Q}}_{\dot{\beta}}\} = 2\sigma^{\mu}_{\alpha\dot{\beta}}\mathbf{P}_{\mu} \tag{9.6}$$

$$[\mathbf{Q}_{\alpha}, \mathbf{P}_{\mu}] = 0 \tag{9.7}$$

$$[\mathbf{Q}_{\alpha}, \mathbf{M}_{\mu\nu}] = i\left(\sigma_{\mu\nu}\right)_{\alpha}^{\ \beta} Q_{\beta} \tag{9.8}$$

Again, the details of eqs. (9.6), (9.7) and (9.8) are not particularly important. The labels

[12]If we allowed these parameters to depend on x we would go towards a theory of gravity. Many attempts in this direction have been made but no satisfactory theory is available yet.

α and $\dot{\beta}$ are spinor indices taking the values 1 and 2, the bar conjugation and the algebra now involves anticommutators and commutators. Another important point to note is that in eqs. (9.6), (9.7) and (9.8) the new generators mix with the other Poincaré generators. This is exactly what we were looking for and what we call a non-trivial extension of the Poincaré algebra.

A theory is supersymmetric, if it is invariant under the group of transformations generated by $\mathbf{P}_\mu$, $\mathbf{M}_{\mu\nu}$ and $\mathbf{Q}$.

Such a theory has some nice properties. First of all, the Hamiltonian is bounded from below and the ground state has zero energy (if susy is not spontaneously broken). Note that the zero energy of the ground state is rather remarkable since in non-supersymmetric theories we usually have to subtract the ground state energy by normal ordering. To understand this we simply have to note that since σ^0 is equal to the unit matrix and $\mathbf{P}$ is the Hamiltonian, eq. (9.6) entails

$$\{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_{\dot{\beta}}\} = 2\delta_{\alpha\dot{\beta}}\mathbf{H} \tag{9.9}$$

From this we conclude for an arbitrary state $|\psi\rangle$

$$\langle\psi|\mathbf{H}|\psi\rangle = \langle\psi|\mathbf{Q}\overline{\mathbf{Q}}|\psi\rangle = \|\overline{\mathbf{Q}}|\psi\rangle\| \geq 0 \tag{9.10}$$

At the same time we see that

$$\langle\psi|\mathbf{H}|\psi\rangle = 0 \quad \Leftrightarrow \quad \overline{\mathbf{Q}}|\psi\rangle = 0 \tag{9.11}$$

which is precisely the condition for susy not to be spontaneously broken (see eq. (5.13)).

Another property of a supersymmetric theory is that the for each bosonic state there is a fermionic state with the same energy, and vice-versa. This follows directly from the fact that the Hamiltonian commutes with $\mathbf{Q}$ (see eq. (9.7)).

9.2 The Supersymmetric Harmonic Oscillator

In this subsection we will consider the simplest supersymmetric model and convince ourselves that this model has indeed all the nice properties we expect.

Let us start with the usual (bosonic) harmonic oscillator. The Hamiltonian is given by

$$H_B = \frac{1}{2}\left(p^2 + \omega_B^2\, x^2\right) \tag{9.12}$$

If we define creation and annihilation operators

$$a \equiv \frac{1}{\sqrt{2\omega_B}}\,(p - i\omega_B x)\,; \qquad a^+ \equiv \frac{1}{\sqrt{2\omega_B}}\,(p + i\omega_B x)\,; \tag{9.13}$$

then the canonical commutation relation $[p, x] = -i$ entails the usual commutation relations for the creation and annihilation operators

$$[a, a^+] = 1; \quad [a, a] = [a^+, a^+] = 0 \tag{9.14}$$

If we write the Hamiltonian eq. (9.12) in terms of the creation and annihilation operators we get

$$H_B = \frac{\omega_B}{2}\left(a^+ a + a\, a^+\right) = \omega_B \left(N_B + \frac{1}{2}\right) \tag{9.15}$$

where we defined the counting operator $N_B \equiv a^+ a$. The energy spectrum of this Hamiltonian (i.e. its eigenvalues) is given by

$$E_{n_B} = \omega_B \left(n_B + \frac{1}{2}\right) \quad \text{with} \quad n_B = 0, 1, 2, 3 \ldots \tag{9.16}$$

A point to note is that the ground-state energy E_0 is $1/2$ and not 0. In a quantum field theory this leads to the problem with the infinite ground-sate energy. This problem is solved by normal ordering.

Let us now repeat these steps for a fermionic harmonic oscillator. We introduce fermionic creation and annihilation operators b and b^+. They satisfy

$$\{b, b^+\} = 1; \quad \{b, b\} = \{b^+, b^+\} = 0 \tag{9.17}$$

These relations correspond to eq. (9.14). However, since we are dealing with fermionic operators now, the commutators are replaced by anticommutators. In analogy to eq. (9.15) we write the Hamiltonian of the fermionic harmonic oscillator as

$$H_F = \frac{\omega_F}{2}\left(b^+ b - b\, b^+\right) = \omega_F \left(N_F - \frac{1}{2}\right) \tag{9.18}$$

where we introduced another counting operator $N_F \equiv b^+ b$. Note that there is a relative minus sign between the $b^+ b$ and $b\, b^+$ term. This sign is due to the fermionic nature of the creation and annihilation operator.

The energy spectrum of this Hamiltonian is given by

$$E_{n_F} = \omega_F \left(n_F - \frac{1}{2}\right) \quad \text{with} \quad n_F = 0, 1 \tag{9.19}$$

Note that contrary to eq. (9.16), n_F can only take the values 0 or 1. This is a reflection of Pauli's exclusion principle in that there cannot be two fermions in the same state.

If we wish we can find an explicit representation of the creation and annihilation operators in terms of Pauli matrices

$$\begin{aligned} b = \sigma_1 - i\sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ b^+ = \sigma_1 + i\sigma_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{9.20}$$

In this representation the Hamiltonian eq. (9.18) is given by

$$H_F = \frac{\omega_F}{2}\,\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.21}$$

and we see that the eigenvalues of H_F are indeed $\pm\omega_F/2$ as given in eq. (9.19).

Now we are ready to combine the fermionic and the bosonic harmonic oscillator. If we just add the two, we do not increase the symmetry of the theory. In order to do this we also have to require $\omega_B = \omega_F \equiv \omega$. Only in this case do we end up with a supersymmetric model. The Hamiltonian then is

$$H \equiv H_B + H_F\Big|_{\omega_B=\omega_F\equiv\omega} = \frac{\omega}{2}\left(a^+\,a + a\,a^+ + b^+\,b - b\,b^+\right) = \omega\left(a^+\,a + b^+\,b\right) \tag{9.22}$$

First of all we see that naively H has an additional symmetry $a \leftrightarrow b$. A state is now determined by two quantum numbers n_B and n_F and the energy spectrum is

$$E_{n_B,n_F} = \omega\,(n_B + n_F) \quad \text{with} \quad n_B = 0,1,2,3\ldots;\ n_F = 0,1 \tag{9.23}$$

Note that the ground-state energy is $E_{0,0} = 0$. Thus as advertised in section 9.1 the ground state has zero energy. This is simply because the bosonic ground state energy $+1/2$ and the fermionic ground state energy $-1/2$ cancel.

The other feature mentioned in section 9.1, namely that the states appear in pairs (a fermionic and a bosonic state) with the same energy can be seen from eq. (9.23). Indeed, the states $|n_B, n_F = 0\rangle$ and $|n_B - 1, n_F = 1\rangle$ have the same energy. Furthermore $|n_B, n_F = 0\rangle$ is a bosonic state (integer spin), whereas $|n_B - 1, n_F = 1\rangle$ is a fermionic state (half-integer spin).

9.3 Supercharges

In this subsection we want to look at the symmetry found in the previous subsection in a somewhat more formal way.

Through the Noether theorem, a symmetry is always related to a conserved current and a conserved charge. Thus, in a supersymmetric theory there is a conserved supercurrent and a conserved supercharge. It is the latter that generates the transformations and we denote it by $\mathbf{Q}$. Since it is conserved it has to commute with the Hamiltonian.

For the supersymmetric harmonic oscillator the supercharge is given by

$$\mathbf{Q}_1 = \sqrt{\omega}\left(a^+\,b + a\,b^+\right); \qquad \mathbf{Q}_2 = i\,\sqrt{\omega}\left(a^+\,b - a\,b^+\right); \tag{9.24}$$

where, as mentioned after eq. (9.6), $\mathbf{Q}$ has a spinor index. We now show that the supercharges as defined in eq. (9.24) have the desired properties. Using the (anti)commutation rules for the creation and annihilation operators, eqs. (9.14) and (9.17), we can compute

$$\begin{aligned}
\{\mathbf{Q}_1, \mathbf{Q}_1\} &= \omega\{a^+ b + a\, b^+, a^+ b + a\, b^+\} \\
&= \omega\{a^+ b, a\, b^+\} + \omega\{a\, b^+, a^+ b\} \\
&= 2\omega \left(a^+ a\,(1 - b^+ b) + (1 + a^+ a)\, b^+ b\right) \\
&= 2\omega(a^+ a + b^+ b) = 2H
\end{aligned} \tag{9.25}$$

In a similar way we can compute the remaining anticommutators of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and we get

$$\{\mathbf{Q}_1, \mathbf{Q}_1\} = \{\mathbf{Q}_2, \mathbf{Q}_2\} = 2\mathbf{H}; \qquad \{\mathbf{Q}_1, \mathbf{Q}_2\} = 0; \tag{9.26}$$

Note that this is in agreement with eq. (9.9). Now we can see that, as promised, the supercharge is conserved.

$$[\mathbf{Q}_1, \mathbf{H}] = \left[\mathbf{Q}_1, (\mathbf{Q}_1)^2\right] = 0 \tag{9.27}$$

$$[\mathbf{Q}_2, \mathbf{H}] = \left[\mathbf{Q}_2, (\mathbf{Q}_2)^2\right] = 0 \tag{9.28}$$

Eqs. (9.27) and (9.28) allow us to see that the states in this theory come in pairs. In fact, let $|\Psi\rangle$ be an eigenstate of $\mathbf{H}$, i.e. $\mathbf{H}|\Psi\rangle = E_\Psi|\Psi\rangle$. Then $\mathbf{Q}_1|\Psi\rangle$ is an eigenstate of $\mathbf{H}$ with the same energy.

$$\mathbf{H}\mathbf{Q}_1|\Psi\rangle = \mathbf{Q}_1\mathbf{H}|\Psi\rangle = \mathbf{Q}_1 E_\Psi|\Psi\rangle = E_\Psi\mathbf{Q}_1|\Psi\rangle \tag{9.29}$$

If $|\Psi\rangle$ is a bosonic state containing n_B bosons and no fermions, then

$$\mathbf{Q}_1|\Psi\rangle = \mathbf{Q}_1|n_B, 0\rangle = \sqrt{\omega}\left(a^+ b + a\, b^+\right)|n_B, 0\rangle = |n_B - 1, 1\rangle \tag{9.30}$$

is a fermionic state with the same energy. Similarly, if $|\Psi\rangle = |n_B, 1\rangle$ is a fermionic state, $\mathbf{Q}_1|\Psi\rangle = |n_B + 1, 0\rangle$ is a bosonic state with the same energy. Thus, the states come indeed as pairs with the same energy, one fermionic and one bosonic.

Of course, the same argument could have been made with $\mathbf{Q}_2$ rather than with $\mathbf{Q}_1$. However, $\mathbf{Q}_2$ acting on a state $|\Psi\rangle$ produces the same state as $\mathbf{Q}_2$ acting on a state $|\Psi\rangle$. Thus, there are not four but only two states with the same energy.

What we have seen is that if we start with the usual bosonic harmonic oscillator and want to make this theory supersymmetric, then we are led to introduce for every bosonic (fermionic) state a fermionic (bosonic) state with the same energy. This is exactly what happens if we want to make the Standard Model supersymmetric. For each boson (fermion) we have to introduce a fermionic (bosonic) partner, thereby doubling[13] the particle spectrum.

[13]Actually, we also have to introduce two Higgs bosons rather than only one. Thus, in the Higgs sector we even quadruple the particle content.

Of course, it is far from clear whether nature has chosen to implement this symmetry. What is clear is that if it has, the symmetry somehow has to be broken since otherwise we should have found for example a selectron (i.e. an 'electron' with spin 0). If at all present, supersymmetry has to be broken such that the superpartners have much larger mass than the particles of the Standard Model. How precisely this is supposed to work is not clear. More importantly, whether nature shares most theorists view and thinks supersymmetry is nice and should be realized is still an open question!! Even though many theorist think it is only a matter of time until supersymmetry will be found, it is only future experiments that will be able to tell.

9.4 Summary

- The only way to extend the Poincaré symmetry in a non-trivial way is to introduce supersymmetry. This involves the introduction of fermionic generators.
- In a supersymmetric theory the spectrum of states consist of pairs of states, one bosonic and one fermionic state with the same energy.
- Supersymmetry guarantees several nice features. The ground state of the theory has zero energy (unless susy is spontaneously broken) and the Hamiltonian is bounded from below.
- At the time of writing this sentence, there is still no experimental evidence whatsoever that supersymmetry is realized in nature.

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