Lecture 14

- Energy in Damped SHM
- Quality Factor
- Forced Oscillations
- Resonance Width

Energy in Damped SHM

- For lightly damped SHM have seen

  $$x(t) = A\exp\left[-\frac{\gamma t}{2}\right]\cos(\omega_d t + \delta)$$

  where $\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

  and $\gamma = \frac{b}{m}$

- We know that, for an undamped SH oscillator, the total energy is $E = \frac{1}{2}kA^2$

Energy in damped SHM cont.

- Allowing for the exponential decrease of the amplitude in lightly damped SHM, the mechanical energy decreases from

  $E_0 = \frac{1}{2} kA^2$

  according to

  $$E \approx \tfrac{1}{2} k\left(A \exp\left[-\frac{\gamma t}{2} \right] \right)^2$$

  $$\approx \tfrac{1}{2} kA^2 \exp[-\gamma t]$$

  $$\approx E_0 \exp[-\gamma t]$$

Quality Factor in SHM

- The quality factor describes the degree of damping. Definition:

  $$Q = \frac{\text{Energy stored in oscillator}}{\text{Energy dissipated per radian}}$$

- Time for 1 rad. oscillation near natural frequency $\omega_0 = \sqrt{k/m}$

  $$\Delta t = \frac{T}{2\pi} = \frac{1}{2\pi} \frac{1}{f} = \frac{1}{\omega_d} \approx \frac{1}{\omega_0}$$

- Mechanical energy lost in that time

  $$\Delta E = \frac{dE}{dt} \Delta t$$

  $$\approx -\frac{1}{\omega_0} E_0 \gamma \exp[-\gamma t]$$

  $$\approx -\frac{\gamma}{\omega_0} E$$

Quality factor in damped SHM cont.

- Hence

$$Q = \frac{E}{|\Delta E|} = \frac{E}{\gamma E / \omega_0} = \frac{\omega_0}{\gamma} = \omega_0 \tau$$

where the damping time

$$\tau = \frac{m}{b} = \frac{1}{\gamma}$$

- A heavily damped system loses energy rapidly and has low Q, lightly damped systems have high Q values.
- Tuning fork, $Q \sim 10^3$.
- Microwave cavity $Q \sim 10^7 ... 10^8$.

Forced Oscillations

- Drive oscillations with force

$$F_e = F_0 \cos(\omega t)$$

- Newton's second law gives

$$\frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x + \frac{F_0}{m}\cos(\omega t)$$

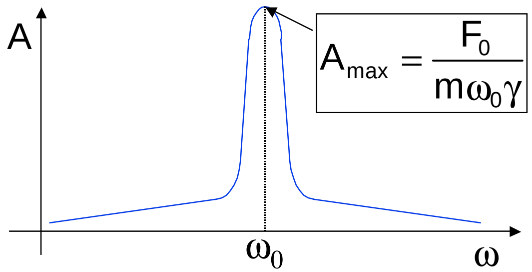
- A solution of this in the steady state (ie. long after starting up system) is $x(t) = A\cos(\omega t + \delta)$ with

$$A = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0{}^2 - \omega^2)^2 + (\omega\gamma)^2}}$$

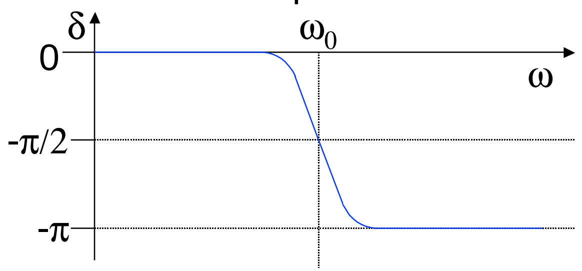
$$\delta = \arctan\left(\frac{\omega\gamma}{\omega^2 - \omega_0{}^2}\right)$$

Forced oscillations cont.

- When damping light see amp. of osc. increases sharply when driving freq. same as natural freq. of system, termed resonance.

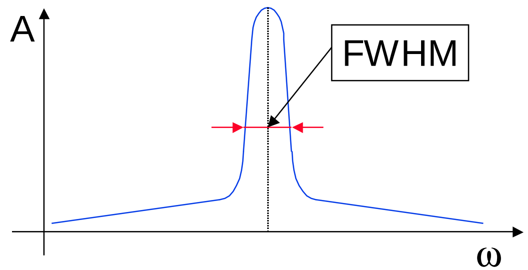


- Phase also flips at resonant freq.



Resonance Width

- Often measured using Full Width at Half Maximum (FWHM).



- Relate FWHM to Q.

$$E \approx \tfrac{1}{2}kA^2$$

$$\approx \frac{1}{2}k\frac{F_0^{\,2}}{m^2}\frac{1}{\left(\omega_0^{\,2}-\omega^2\right)^2+(\omega\gamma)^2}$$

$$\approx \frac{1}{2}\frac{F_0^{\,2}}{m}\frac{\omega_0^{\,2}}{(\omega_0+\omega)^2(\omega_0-\omega)^2+(\omega\gamma)^2}$$

Resonance width cont.

- Near resonance $\omega \approx \omega_0$

$$\omega_0 + \omega \approx 2\omega_0$$

- We get

$$E \approx \frac{1}{2}\frac{F_0^{\,2}}{m}\frac{\omega_0^{\,2}}{4\omega_0^{\,2}(\omega_0 - \omega)^2 + (\omega_0\gamma)^2}$$

$$\approx \frac{1}{8}\frac{F_0^{\,2}}{m}\frac{1}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

- This is the resonance curve or Lorentzian. The amp. decreases by factor 2 from peak at $\omega = \omega_0 \pm \gamma/2$
- Hence FWHM given by $\gamma = \omega_0/Q$ or $Q = \omega_0/\gamma$
- Q sometimes defined w.r.t. width of resonance using this equation.