Lecture 12

- Private Study Topics
- Oscillations
- Simple Harmonic Motion
  - General Solution
  - Energy Considerations

Private Study Topics

- Equilibrium
  - Balance of forces
  - Balance of torques
  - Centre of gravity
  - Indeterminate Structures
- Elasticity
  - Stress and strain
  - Tension
  - Compression

    (Young's modulus)
  - Shearing (shear modulus)
  - Hydraulic compression (bulk modulus)
- See eg. H, R & W, Chapt. 13.

Oscillations

- Repetitive motion, eg. vibrating strings, pendula, vibrating molecules conveying sound waves etc.
- Oscillations occur when a system in stable equilibrium is slightly disturbed.
- Condition for stable equilibrium is minimum of potential U(x), say at position x_0.
- Force F is then

$$F = -\left.\frac{\partial U(x)}{\partial x}\right|_{x_0}$$

$$= 0$$

Oscillations cont.

- Study potential for small disturbances, ie. near $x=x_0$. Use Taylor's expansion:

$$U(x) = U(x_0) + (x - x_0)\left.\frac{\partial U(x)}{\partial x}\right|_{x_0} +$$

$$\frac{(x - x_0)^2}{2!}\left.\frac{\partial^2 U(x)}{\partial x^2}\right|_{x_0} +$$

$$\frac{(x - x_0)^3}{3!}\left.\frac{\partial^3 U(x)}{\partial x^3}\right|_{x_0} + \cdots$$

- Now second term in expansion is zero, and third term much bigger than fourth as $(x-x_0)$ is small, so

$$U(x) \approx U(x_0) + \frac{(x - x_0)^2}{2}\left.\frac{\partial^2 U(x)}{\partial x^2}\right|_{x_0}$$

Oscillations cont.

- The resulting force is

$$F(x) \approx -\frac{\partial}{\partial x}U(x_0) - \frac{\partial}{\partial x}\frac{(x-x_0)^2}{2}\left.\frac{\partial^2 U(x)}{\partial x^2}\right|_{x_0}$$

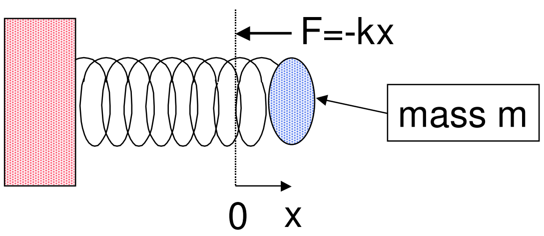
- The first term is zero so we are left with

$$F(x) \approx -(x-x_0)\left.\frac{\partial^2 U(x)}{\partial x^2}\right|_{x_0}$$

- We see that any potential, for small displacements from stable equilibrium, leads to a restoring force proportional to the displacement, eg. complex intermolecular potential gives Hooke's Law for springs

Simple Harmonic Motion

- Given previous result, let us examine motion in which force proportional to displacement from equilibrium in more detail. Such motion termed Simple Harmonic Motion.
- Define origin at position of equilibrium, then SHM force is $F = -kx$



Simple harmonic motion cont.

- Using Newton's Second Law
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{F}{m} = -\frac{kx}{m}$$
- Now must solve homogeneous 2nd order differential equation
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$
- "Mechanics", by Smith & Smith describes how to do this, we merely note here that the general solution may be written
$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t + \delta\right)$$

Simple harmonic motion cont.

- We see force law typical of oscillations resulting from small displacements from stable equilibrium leads to sinusoidal oscillations.
- The period of the oscillations may be found by remembering that sine functions repeat every 2π. Hence one cycle occurs in a time T such that
$$\sqrt{\frac{k}{m}}(t+T) + \delta = \sqrt{\frac{k}{m}}t + \delta + 2\pi$$
$$\sqrt{\frac{k}{m}}T = 2\pi$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$

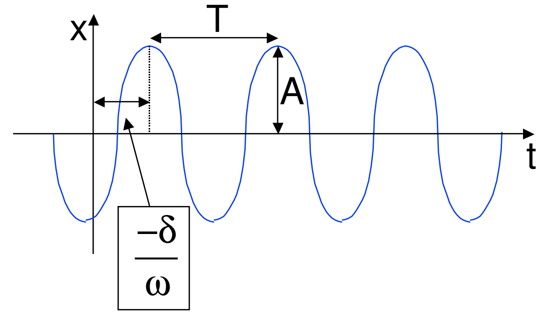
Simple harmonic motion cont.

- Frequency is
  $$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
- Units Hz or s^{-1}.
- Angular frequency
  $$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$
- Units rad^{-1}.
- Using above may write differential equation for simple harmonic motion
  $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$
- With solution
  $$x(t) = A\cos(\omega t + \delta)$$

Simple harmonic motion cont.

- Look at equation for position of particle undergoing SHM

$$x(t) = \overbrace{A}^{\text{amplitude}} \cos(\underbrace{\underbrace{\omega}_{\text{ang. freq.}} t + \overbrace{\delta}^{\text{phase const.}}}_{\text{phase}})$$



Simple harmonic motion cont.

- Differentiating expression for position w.r.t. time gives velocity

$$v(t) = \frac{d}{dt}x(t) = -A\omega\sin(\omega t + \delta)$$

- Differentiating again gives acceleration

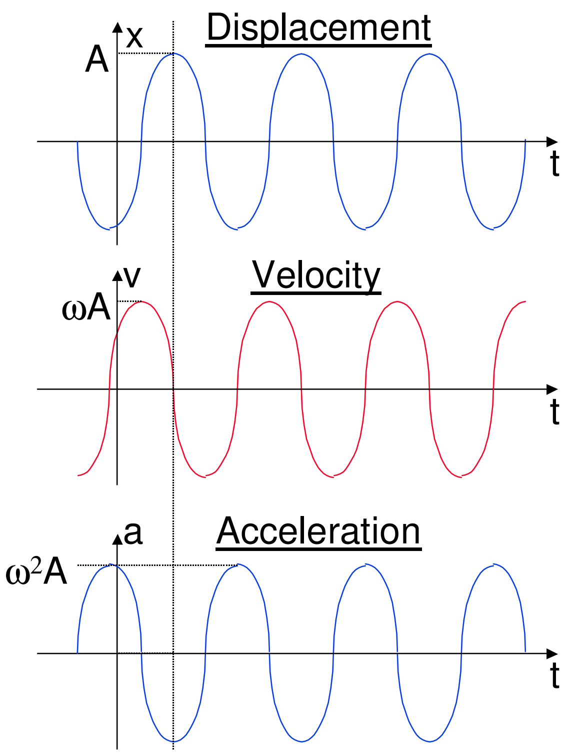
$$a(t) = \frac{d}{dt}v(t) = -A\omega^2\cos(\omega t + \delta)$$

- Recall expression for position

$$a(t) = \frac{d^2}{dt^2}x(t) = -\omega^2 x(t)$$

- This proves that our expression for x(t) is indeed a solution of the SHM differential equation.

Simple harmonic motion cont.



Energy Considerations

- We defined SHM as motion with restoring force proportional to displacement, ie.
  $F = -kx$
- Either using results from section on oscillations, or by integration we can obtain corresponding potential

$$F(x) = -\frac{dU(x)}{dx}$$

$$\Rightarrow U(x) = -\int F(x)dx + U_0$$

$$= \int kx\,dx = \frac{kx^2}{2} + U_0$$

$$= \frac{A^2k}{2}\cos^2(\omega t + \delta) + U_0$$

Energy considerations cont.

- We know also that the kinetic energy is
  $$K = \frac{mv^2}{2} = \frac{A^2m\omega^2}{2}\sin^2(\omega t + \delta)$$
- We see that the total energy (setting U_0=0) is
  $$E = K + U = \frac{A^2m\omega^2}{2}\sin^2(\omega t + \delta) + \frac{A^2k}{2}\cos^2(\omega t + \delta)$$

Energy considerations cont.

- Now using $\omega^2 = \frac{k}{m}$

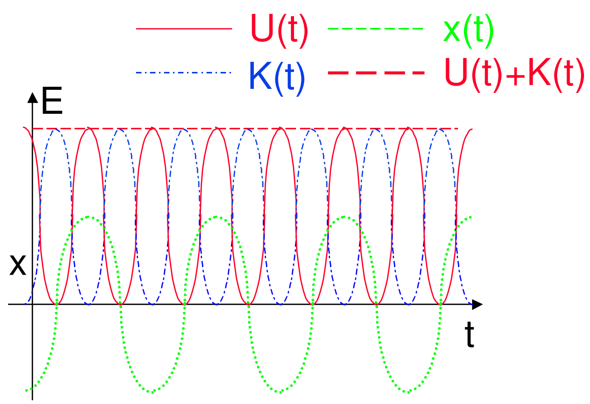
$$E = \frac{A^2k}{2}(\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta))$$

$$= \frac{A^2k}{2}$$

- Total mechanical energy is constant, energy continuously shifting between kinetic and potential forms. In practice mechanical energy will always be dissipated, leads to damped SHM.

Energy considerations cont.

Energy as function of time



Energy as a function of position