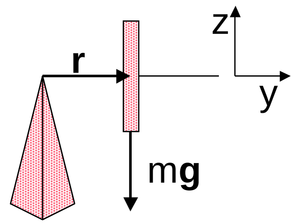
Lecture 10

- Precession of Gyroscope
- Wheels
- Rolling
- More Moments of Inertia

Precession of Gyroscope

- Gyroscope not spinning, falls if released from horizontal.



$$\mathbf{r}_0 = (0, a, 0)$$
$$\vec{\tau}_0 = \mathbf{r}_0 \times \mathbf{F}_0 = (-mga, 0, 0)$$
$$\mathbf{L}_0 = (0, 0, 0)$$

- Newton's 2nd Law

$$\vec{\tau} = \frac{d}{dt}\mathbf{L} \Rightarrow \vec{\tau}dt = d\mathbf{L}$$

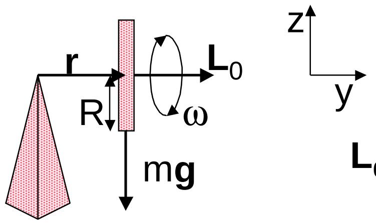
- **L** after time dt (Euler's method).

$$\mathbf{L}_1 = \mathbf{L}_0 + d\mathbf{L}_0 = \mathbf{L}_o + \vec{\tau}_0 dt = (0,0,0) + (-mga, 0, 0)dt = (-mga\,dt, 0, 0)$$

- Gyroscope starts to rotate about x axis (falls)

Precession of gyroscope cont.

- If gyroscope spinning rapidly



$$\mathbf{L_0} = I\vec{\omega}_0 = I(0,\omega,0)$$

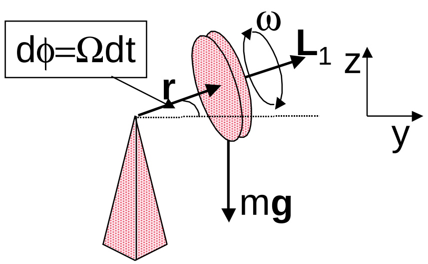
- **L** after time dt now

$$\mathbf{L_1} = (0, I\omega, 0) + (-mga, 0, 0)dt = (-mga\,dt, I\omega, 0)$$

- If gyroscope were to fall, **L** would acquire z component due to spin of gyroscope. No z component, so it doesn't fall!

Precession of gyroscope cont.

- Where does x component of **L** come from? Must be from spin.



- |**L**| due to spin ($\Omega \ll \omega$) so dϕ such that spin along x is -mga dt ie.

$$I\omega \sin d\phi = mga\,dt$$

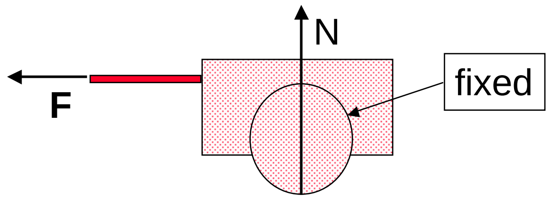
$$\frac{d\phi}{dt} = \frac{mga}{I\omega}$$

- Using $I = \frac{1}{2}mR^2$, $\Omega = \frac{d\phi}{dt} = \frac{2ga}{R^2\omega}$

  is the precession rate.

Wheels

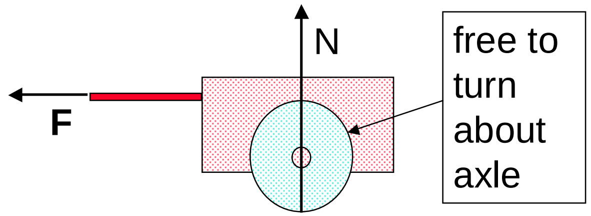
- Work out force necessary to drag sled.



- To overcome kinetic friction require $F_s > f_k = \mu_k N$
- What advantage does using primitive wheel (no bearings) bring?

Wheels cont.

- Calc. force necessary to move cart, rad. of wheel R, of axle r.



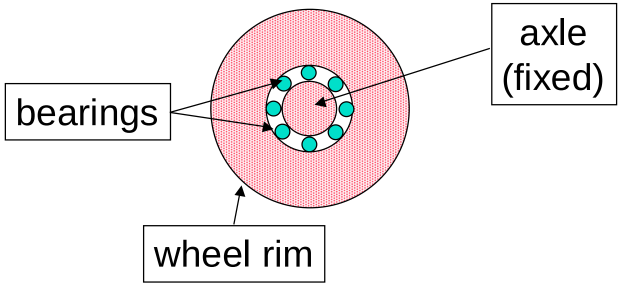
- Force needed to move cart is force that produces torque needed to turn wheel.
- Must overcome torque due to friction $\tau_f = f_k r = \mu_k N r$
- That is need torque $\tau = F_w R > \tau_f$ so $F_w > \dfrac{\mu_k N r}{R}$

$$F_w = \frac{r}{R} F_s$$

Wheels cont.

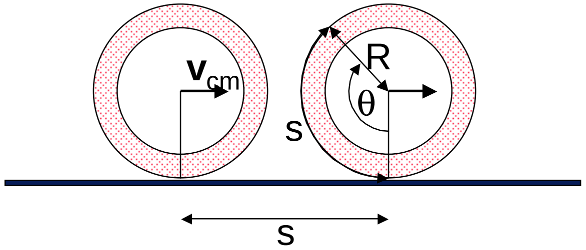
- Have simplified as we have ignored "rolling" friction.

  $f_r = \mu_r N$
- Typically coeff. of rolling friction is factor 10 smaller than coeff. of kinetic friction.
- If introduce bearings, necessary force decreases further, must then overcome only rolling friction.



Rolling

- Consider motion of wheel.



- No slipping so

$$s = R\theta$$

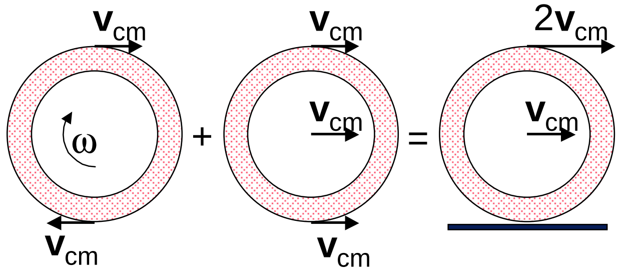
$$\Rightarrow \frac{ds}{dt} = \frac{d}{dt}R\theta = R\frac{d\theta}{dt}$$

$$v_{cm} = R\omega$$

$$\omega = \frac{v_{cm}}{R}$$

Rolling cont.

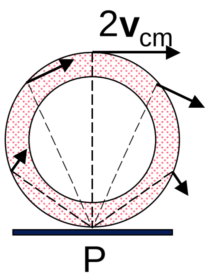
- Consider as rotation + translation



- Calculate total K.E. in this picture

$$K = K_R + K_T = \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} M v_{cm}^{\ 2}$$

- Now think as pure rotation



$$\omega = \frac{2v_{cm}}{2R} = \frac{v_{cm}}{R}$$

Angular velocity same in both pictures!

Rolling cont.

- Calculate K.E. in this picture

$$K' = K'_R = \tfrac{1}{2} I_P \omega^2$$

- Use parallel axis theorem

$$I_P = I_{cm} + MR^2$$

$$K'_R = \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} MR^2\omega^2$$

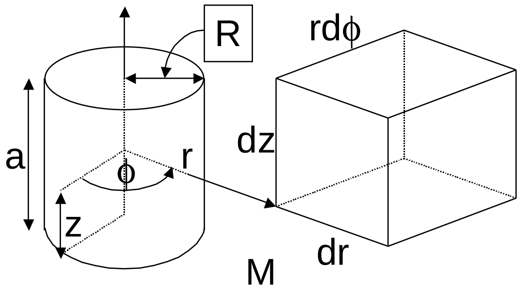
$$= \tfrac{1}{2} I_{cm}\omega^{\ 2} + \tfrac{1}{2} M v_{cm}^{\ 2}$$

$$= K_R + K_T$$

- K.E. same regardless of picture used.

More Moments of Inertia

- Solid cylinder or disc about axis



$$\rho = \frac{M}{\pi R^2 a}$$

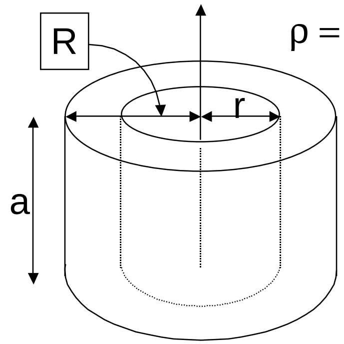
$$I = \rho \int_V r^2 dV$$

$$= \rho \int_0^a \int_0^{2\pi} \int_0^R r^2 dr\, r d\phi\, dz$$

$$= 2\pi a \rho \int_0^R r^3 dr = 2\pi a \rho \frac{R^4}{4} = \frac{MR^2}{2}$$

Moments of inertia cont.

- Hollow cylinder about axis



$$\rho = \frac{M}{\pi (R^2 - r^2) a}$$

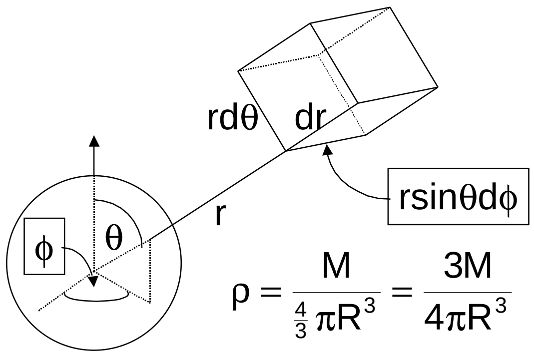
$$I = \rho \int_0^a \int_0^{2\pi} \int_r^R r^2 dr\, r d\phi\, dz$$

$$= 2\pi \rho a \int_r^R r^3 dr = \frac{\pi \rho a}{2} (R^4 - r^4)$$

$$= \frac{M(R^2 + r^2)(R^2 - r^2)}{2(R^2 - r^2)} = \frac{M(R^2 + r^2)}{2}$$

Moments of inertia cont.

- Sphere radius R about centre



$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$$I = \rho \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (r\sin\theta)^2 \, dr \, r d\theta \, r\sin\theta d\phi$$

$$= 2\pi\rho \int_0^{\pi}\int_0^{R} r^4 \sin^3\theta \, dr \, d\theta$$

Moments of inertia cont.

Do integral over θ first. Using

$$\int \sin^n\theta d\theta = -\frac{\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta d\theta$$

we find

$$I = 2\pi\rho\int_0^R r^4 \left(-\frac{\sin^2\theta\cos\theta}{3}\Bigg|_0^{\pi} + \frac{2}{3}\int_0^{\pi}\sin\theta d\theta \right) dr$$

$$= 2\pi\rho\int_0^R r^4 \left(-\frac{2\cos\theta}{3}\right)\Bigg|_0^{\pi} dr = \frac{8}{3}\pi\rho\int_0^R r^4 dr$$

$$= \frac{8}{3}\pi\rho\frac{R^5}{5} = \frac{2}{5}MR^2$$