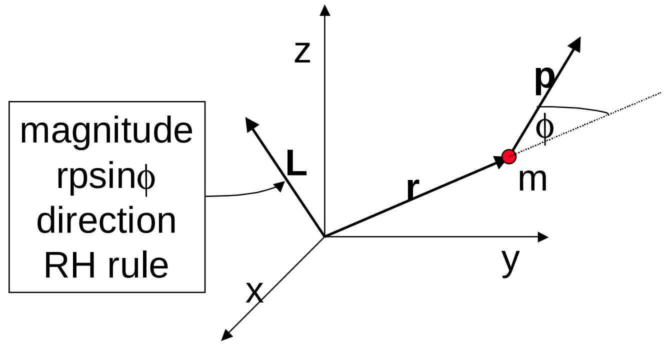
Lecture 9

- Angular Momentum
- Newton's Second Law for Rotation in Vector Form
- Angular Momentum for Systems of Particles
- Conservation of Angular Momentum

Angular Momentum

- Consider a particle with momentum **p**=m**v** and position vector **r**. Its angular momentum is

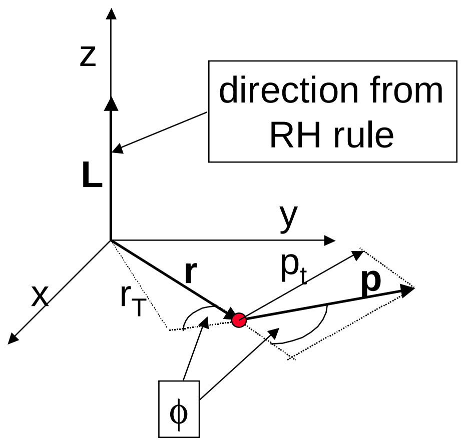
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
$$= m\mathbf{r} \times \mathbf{v} \text{ units kg m}^2 \text{ s}^{-1} \text{ or Js.}$$



Note, no requirement that motion be "circular"!

Angular momentum cont.

- Gain some insight into angular momentum by considering motion in (x,y) plane.



$$|\mathbf{L}| = rp_T = r_T p$$
$$= rp\sin\phi \text{ (c.f. torque).}$$

Newton's Second Law Revisited

- Obtain Newton's 2nd Law for rotational motion from that for linear motion and definition of angular momentum.

$$\mathbf{L} = \mathbf{r}\times\mathbf{p}$$
$$\frac{d}{dt}\mathbf{L} = \frac{d\mathbf{r}}{dt}\times\mathbf{p} + \mathbf{r}\times\frac{d\mathbf{p}}{dt}$$
$$= \mathbf{v}\times\mathbf{p} + \mathbf{r}\times\frac{d\mathbf{p}}{dt}$$

- Using collinearity of **v** and **p**

$$\frac{d}{dt}\mathbf{L} = \mathbf{r}\times\frac{d\mathbf{p}}{dt}$$

- Newton's 2nd Law for linear motion is

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Newton's second law cont.

- Substituting gives

$$\frac{d}{dt}\mathbf{L} = \mathbf{r} \times \left(\sum \mathbf{F}\right)$$
$$= \sum \mathbf{r} \times \mathbf{F}$$
$$= \sum \vec{\tau}$$

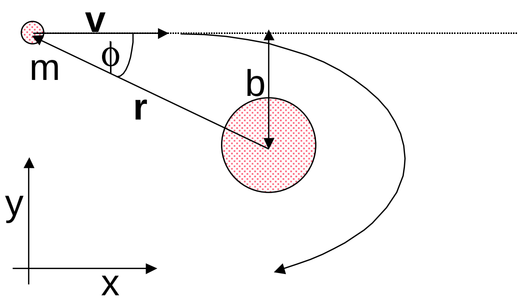
- Newton's Second Law for rotation is thus

$$\sum \vec{\tau} = \frac{d}{dt}\mathbf{L}$$

- No torque applied means angular momentum conserved.

Angular Momentum, an Example

- Angular momentum in collision of two particles in (x,y) plane. Assume target very heavy.



- Initial angular momentum

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v}$$
$$= (0,0,-mrv\sin\phi)$$
$$= (0,0,-mvb)$$

Angular momentum, an example cont.

- Conservation of angular momentum implies:
  - x and y components of angular momentum always zero, i.e. motion confined to plane.
  - z component always -mvb, i.e. speed must increase as perpendicular distance to target decreases (cf. Kepler's Laws, later in course).

Systems of Particles

- Total angular momentum of system of particles is

$$\mathbf{L}_{Tot} = \sum_i \mathbf{L}_i$$

- Newton's 2nd Law for system

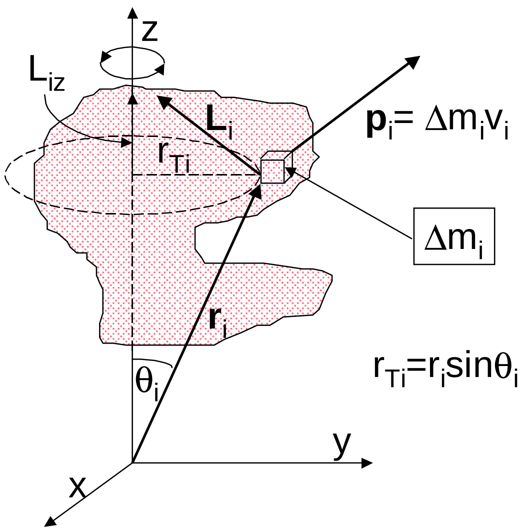
$$\frac{d}{dt}\mathbf{L}_{Tot} = \sum \frac{d}{dt}\mathbf{L}_i = \sum \vec{\tau}_i$$

- Equal and opposite torques between particles in body cancel so we have

$$\sum_{ext} \vec{\tau}_i = \frac{d}{dt}\mathbf{L}$$

Angular Momentum of Rigid Body

- Consider body rotating about z axis with angular velocity ω.



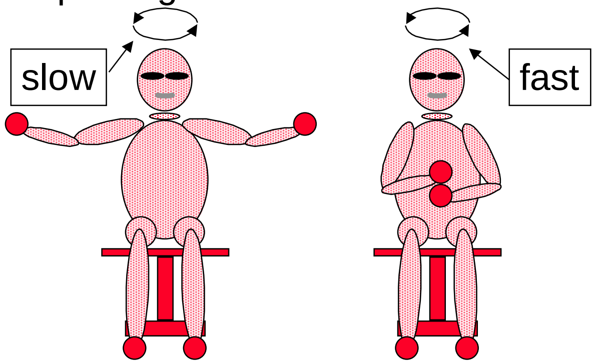
$$L_{Tot\,z} = \sum_i L_{iz} = \sum_i \Delta m_i v_i r_{Ti} = \sum_i \Delta m_i (\omega r_{Ti}) r_{Ti}$$
$$= \omega \sum_i \Delta m_i r_{Ti}^{\;2}$$

Angular momentum of rigid body cont.

- Recall $I_z = \sum m_i r_{Ti}^{\;2}$ we see that $L_z = I_z \omega_z$
- Differentiating gives Newton's 2nd Law for the rigid body
$$\tau_{ext\,z} = \frac{d}{dt} I_z \omega_z$$
- If no external torques, angular momentum conserved.
- Note that both I_z and ω_z may change with time (eg. ice-skater doing pirouette)
- If I_z constant then above becomes
$$\tau_{ext\,z} = I_z \frac{d}{dt} \omega_z$$
$$= I_z \alpha_z$$

Conservation of Angular Momentum

- "Spinning Volunteer".



- Initial angular momentum $I_i\omega_i$, final angular momentum $I_f\omega_f$. From conservation of angular momentum

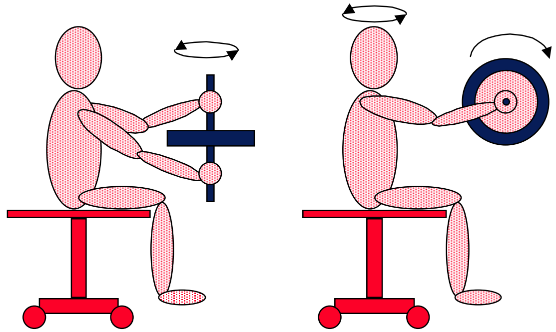
$$I_f\omega_f = I_i\omega_i$$

$$\Rightarrow \omega_f = \frac{I_i}{I_f}\omega_i$$

- Pulling in arms reduces I, so ω increases.

Conservation of angular momentum cont.

- "Volunteer with wheel"



- Change orientation of spinning wheel, volunteer rotates, see vector nature of angular momentum. E.g. wheel axis initially vertical, angular momentum vertical. Change to horizontal (volunteer must apply torque!) then volunteer's rotation ensures angular momentum of system (no external torque) conserved.