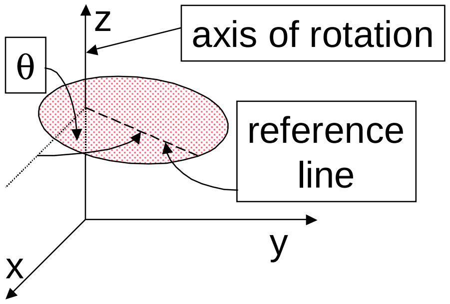
Lecture 7

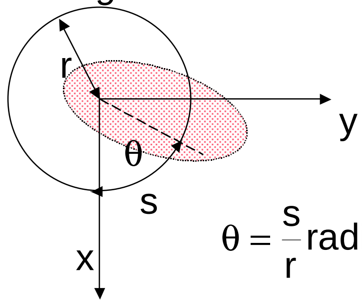
- Rotational variables:
  - Angular position and displacement.
  - Vectors, large and small angular displacements.
  - Angular velocity and acceleration.
- Constant angular acceleration and rotational motion.
- Kinetic energy and the moment of inertia.
- Calculation of the moment of inertia.

Rotational Variables

- Consider rotations of rigid body about a fixed axis.



Looking from above



$$\theta = \frac{s}{r} \text{ radians}$$

Rotational variables cont.

- Units radians, dimensionless, (ratio of two lengths).

$$2\pi \text{ rad} = 360^\circ = 1 \text{ rev}$$

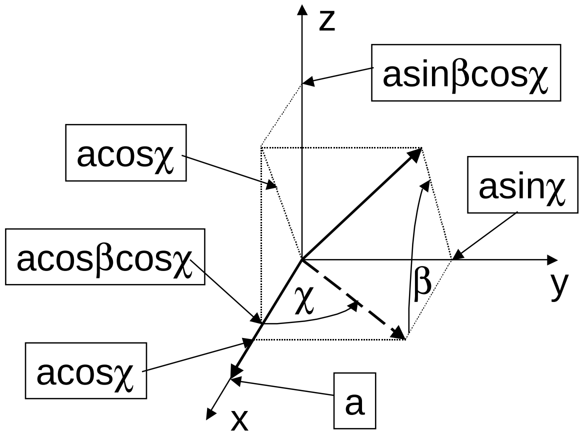
$$1 \text{ rad} = 57.3^\circ = 0.159 \text{ rev}$$

$$1^\circ = 17.5 \text{ mrad} = 0.00278 \text{ rev}$$

- Angular displacement
  $\Delta\theta = \theta_2 - \theta_1$
- Is angular displacement a vector?
- No! Addition of rotations is not commutative and definition of vectors requires that addition be commutative.

Angular Displacement not Vector

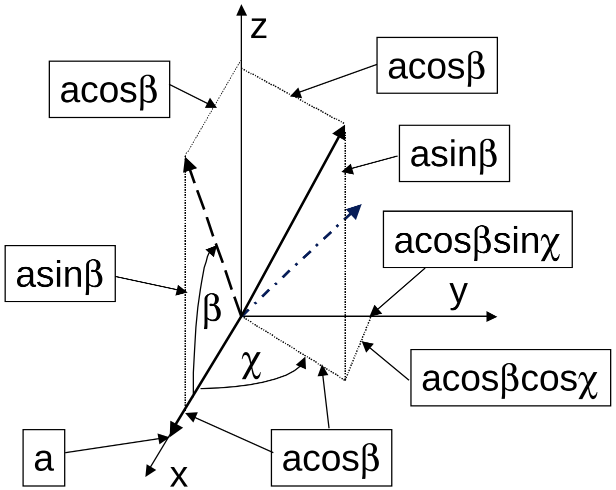
- Proof in pictures.
  Rotate about z then about y



$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a\cos\chi \\ a\sin\chi \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a\cos\beta\cos\chi \\ a\sin\chi \\ a\sin\beta\cos\chi \end{pmatrix}$$

Angular displacement not vector cont.

Rotate about y *then about* z



$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a\cos\beta \\ 0 \\ a\sin\beta \end{pmatrix} \rightarrow \begin{pmatrix} a\cos\beta\cos\chi \\ a\cos\beta\sin\chi \\ a\sin\beta \end{pmatrix}$$

Angular displacement not vector cont.

- Rotation about z then y gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos\beta\cos\chi \\ a\sin\chi \\ a\sin\beta\cos\chi \end{pmatrix}$$

- Rotation about y then z gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos\beta\cos\chi \\ a\cos\beta\sin\chi \\ a\sin\beta \end{pmatrix}$$

- These are not the same! QED.
- Look now at result for small angles.

Small Displacements

- Consider small angular displacements $\Delta\beta$ and $\Delta\chi$ about y and z axes. Use approximations:

$$\sin \Delta\beta \approx \Delta\beta$$

$$\cos \Delta\beta \approx 1 - \tfrac{1}{2}\Delta\beta^2 \approx 1$$

- Rotate about z then y

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos\Delta\beta\cos\Delta\chi \\ a\sin\Delta\chi \\ a\sin\Delta\beta\cos\Delta\chi \end{pmatrix} = \begin{pmatrix} a \\ a\,\Delta\chi \\ a\,\Delta\beta \end{pmatrix}$$

- Rotate about y then z

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos\Delta\beta\cos\Delta\chi \\ a\cos\Delta\beta\sin\Delta\chi \\ a\sin\Delta\beta \end{pmatrix} = \begin{pmatrix} a \\ a\,\Delta\chi \\ a\,\Delta\beta \end{pmatrix}$$

- Same, so small angles vectors

Angular Velocity and Acceleration

- Define angular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- Units rad s^{-1}, rev/s, rpm.
- Angular displacement “small” so angular velocity is a vector.
- Same applies to angular acceleration.

$$\alpha = \frac{d\omega}{dt}$$

- Units rad s^{-2}, rev/s^2.
- Derive equations for motion with constant angular acceleration, analogous to equations for motion with constant linear acceleration.

Motion with Constant Acceleration

- Starting point

$$\alpha = \frac{d\omega}{dt}$$

$$\int_0^t \alpha dt' = \int_{\omega_0}^{\omega} d\omega'$$

$$\alpha t = \omega - \omega_0$$

$$\omega = \omega_0 + \alpha t \qquad (1)$$

- Integrate again

$$\int \alpha t dt = \int \omega - \omega_0 dt$$

$$\frac{\alpha t^2}{2} = \theta - \omega_0 t$$

$$\theta = \omega_0 t + \frac{\alpha t^2}{2} \qquad (2)$$

Motion with Constant Acceleration cont.

- Calc. t from (1) and subst. in (2)

$$\omega^2 = \omega_0{}^2 + 2\alpha\theta \qquad (3)$$

- Calc. α from (1) and subst. in (2)

$$\theta = \frac{\omega_0 + \omega}{2} t \qquad (4)$$

- Calc ϖ_0 from (1) and subst. in (2)

$$\theta = \omega t - \frac{\alpha t^2}{2} \qquad (5)$$

Constant Acceleration, Linear and Angular

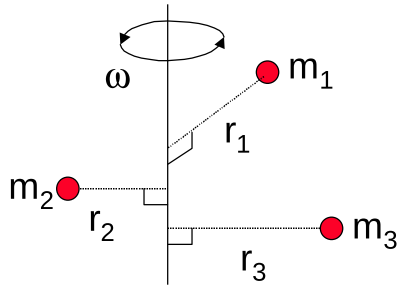
- Compare equations of motion:

	Angular	Linear
Accel.	α	a
Final vel.	ω	v
Initial vel.	ω_0	u or v_0
Final pos.	θ	s
Time	t	t

$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{\alpha t^2}{2}$	$s = v_0 t + \frac{at^2}{2}$
$\omega^2 = \omega_0{}^2 + 2\alpha\theta$	$v^2 = v_0{}^2 + 2as$
$\theta = \frac{\omega_0 + \omega}{2} t$	$s = \frac{v_0 + v}{2} t$
$\theta = \omega t - \frac{\alpha t^2}{2}$	$s = vt - \frac{at^2}{2}$

Kinetic Energy and the Moment of Inertia

- What is analogue of mass in rotational motion?
- Consider K.E. of many particles rotating around axis.



- K.E. given by

$$K = \tfrac{1}{2}\sum m_i v_i^2$$
$$= \tfrac{1}{2}\sum m_i r_i^2 \omega^2$$

- Note, r_i is perpendicular distance from axis to i^{th} particle.

K.E. and the moment of inertia cont.

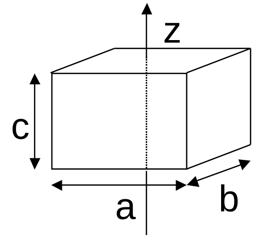
- Recall for translational K.E. of body of total mass M
  $K_T = \frac{1}{2} M v_{cm}^2$
- So analogue of mass for rotation may be seen to be the moment of inertia:
  $I = \sum m_i r_i^2$
- Rotational K.E. is then
  $K_R = \frac{1}{2} I \omega^2$
- Extend definition of moment of inertia to continuous bodies

$$I = \int_M r^2\, dm = \int_V r^2 \frac{dm}{dV}\, dV$$
$$= \int_V r^2 \rho\, dV$$

Calculating Moments of Inertia

- Uniform rectangular prism of density ρ

$$M = abc\rho$$



$$I = \rho \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2)\, dx\, dy\, dz$$
$$= \rho \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2)\, z \Big|_{-\frac{c}{2}}^{\frac{c}{2}}\, dx\, dy$$
$$= c\rho \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2)\, dx\, dy$$

Calculating moments of inertia cont.

$$I = c\rho \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 y + \frac{y^3}{3} \Bigg|_{-\frac{b}{2}}^{\frac{b}{2}} dx$$

$$= c\rho \int_{-\frac{a}{2}}^{\frac{a}{2}} bx^2 + \frac{b^3}{12}\, dx$$

$$= bc\rho \left(\frac{x^3}{3} + \frac{b^2 x}{12} \right) \Bigg|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= bc\rho \left(\frac{a^3}{12} + \frac{ab^2}{12} \right)$$

$$= \frac{abc\rho(a^2 + b^2)}{12} = \frac{M(a^2 + b^2)}{12}$$