ECE 3313
ELECTROMAGNETICS I

Electromagnetics (EM) - the study of electric and magnetic phenomena.

     A knowledge of the fundamental behavior of *electric* and *magnetic* fields is necessary to understand the operation of such devices as resistors, capacitors, inductors, diodes, transistors, transformers, motors, relays, transmission lines, antennas, waveguides, optical fibers and lasers.

     All electromagnetic phenomena are governed by a set of four equations known as *Maxwell's equations*.

Maxwell's Equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \cdot \boldsymbol{D} = \rho_v$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$\boldsymbol{E}$ - electric field intensity
$\boldsymbol{H}$ - magnetic field intensity
$\boldsymbol{D}$ - electric flux density
$\boldsymbol{B}$ - magnetic flux density
$\boldsymbol{J}$ - current density
ρ_v - volume charge density

ECE 3313 Electromagnetics I
     Electrostatic fields, magnetostatic fields, electromagnetic fields, electromagnetic waves, transmission lines.

ECE 3323 Electromagnetics II
     Waveguides, fiber optics, antennas, electromagnetic compatibility, analytical and computational electromagnetics.

Vector Algebra

The quantities of interest appearing in Maxwell's equations along with other quantities encountered in the study of EM can almost always be classified as either a *scalar* or a *vector* (*tensors* are sometimes encountered in EM but will not be covered in this class).

Scalar - a quantity defined by magnitude only.
[examples: distance (x), voltage (V), charge density (ρ_v), etc.]

Vector - a quantity defined by magnitude and direction.
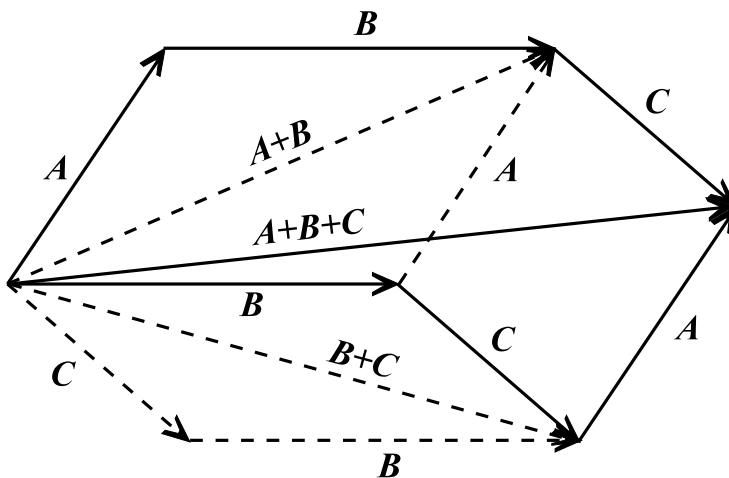[examples: velocity (v), current (I), electric field (E), etc.]

Note that vectors are denoted with boldface letters. The magnitude of a vector may be a real-valued scalar or a complex-valued scalar (phasor).
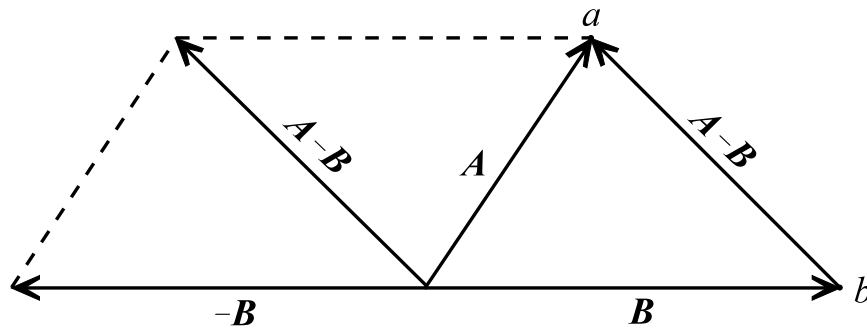
Vector Addition (Parallelogram Law)
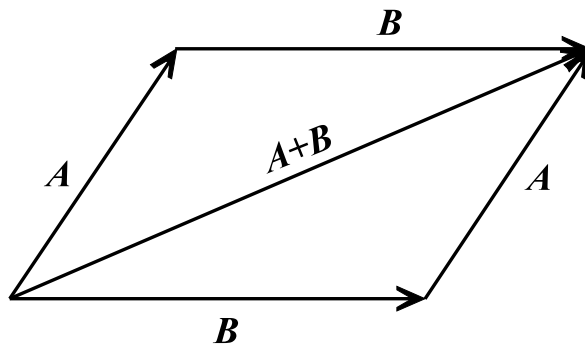
Commutative Law
$A+B = B+A$

Associative Law
$(A+B)+C = A+(B+C)$

Vector Subtraction

$$A - B = A + (-B)$$

Note:
    (1)   The magnitude of the vector $A - B$ is the separation distance d between the points a and b located by the vectors A and B, respectively [$d = |A - B| = |B - A|$].

    (2)   The vector $A - B$ is the vector pointing from b (origination point) to a (termination point).

Multiplication and Division By a Scalar

$$a(A + B) = aA + aB \qquad \text{(Distributive law)}$$

$$\frac{A + B}{a} = \frac{1}{a}A + \frac{1}{a}B$$

Coordinate Systems

A coordinate system defines points of reference from which specific vector directions may be defined. Depending on the geometry of the application, one coordinate system may lead to more efficient vector definitions than others. The three most commonly used coordinate systems used in the study of electromagnetics are *rectangular* coordinates (or *cartesian* coordinates), *cylindrical* coordinates, and *spherical* coordinates.

Rectangular Coordinates
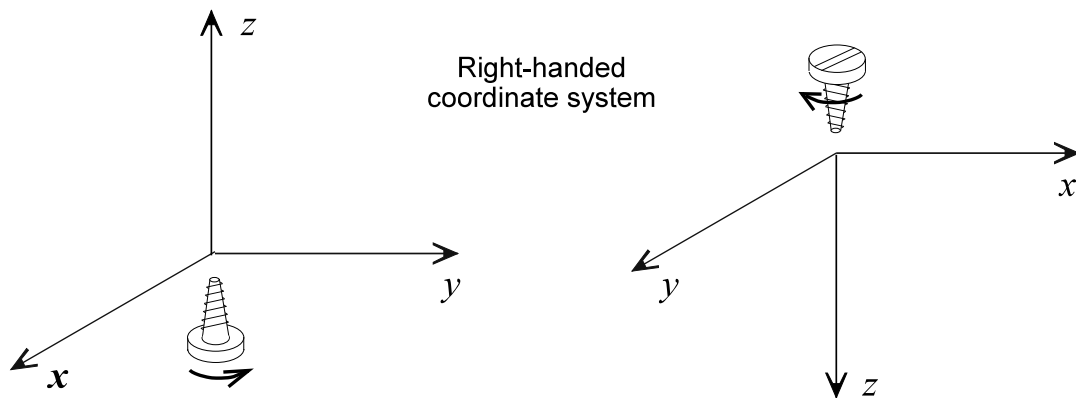
The rectangular coordinate system is an *orthogonal* coordinate system with coordinate axes defined by x, y, and z. The coordinate axes in an orthogonal coordinate system are mutually perpendicular. By convention, we choose to define rectangular coordinates as a *right-handed* coordinate system. This convention ensures that the three coordinate axes are always drawn with the same orientation no matter how the coordinate system may be rotated. If we position a right-handed screw normal to the plane containing the x and y axes, and rotate the screw in the direction of the x axis rotated toward the y axis, the direction that the screw advances defines the direction of the z axis in a right-handed coordinate system.

Component Scalars and Component Vectors
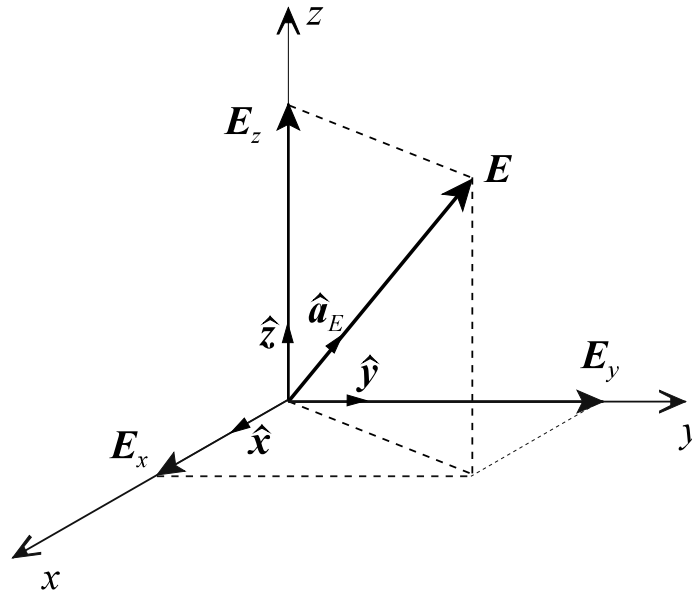
Given an arbitrary vector E in rectangular coordinates, the vector E can be described (using vector addition) as the sum of three *component vectors* that lie along the coordinate axes.

$$E = E_x + E_y + E_z$$

Component Vectors

The component vectors can be further simplified by defining *unit vectors* along the coordinate axes: $\hat{x}, \hat{y}$ and $\hat{z}$. Each of these unit vectors have magnitudes of unity and directions parallel to the respective coordinate axis. The component vectors can be written in terms of the unit vectors as

$$E_x = |E_x|\,\hat{x} = E_x\,\hat{x}$$

$$E_y = |E_y|\,\hat{y} = E_y\,\hat{y}$$

$$E_z = |E_z|\,\hat{z} = E_z\,\hat{z}$$

$$E = E_x + E_y + E_z = E_x\,\hat{x} + E_y\,\hat{y} + E_z\,\hat{z}$$

Component Scalars

Thus, using *component scalars*, any rectangular coordinate vector can be uniquely defined using three scalar quantities that represent the magnitudes of the respective component vectors.

To define a unit vector in the direction of E, we simply divide the vector by its magnitude.

$$\hat{a}_E = \frac{E}{|E|} = \frac{E_x \hat{x} + E_y \hat{y} + E_z \hat{z}}{\sqrt{E_x^2 + E_y^2 + E_z^2}} \qquad \left(\begin{array}{c} \text{unit vector in the} \\ \text{direction of } E \end{array} \right)$$

where the magnitude of E is the diagonal of the rectangular volume formed by the three component scalars.

Example (Unit vector)

Given the vector $E = (x + y)\hat{x} + 3\hat{y} + z^2 \hat{z}$, determine the unit vector in the direction of E at the rectangular coordinate location of $(1,1,1)$.

$$E_x = x + y$$

$$E_y = 3$$

$$E_z = z^2$$

Note that two of the component scalars are functions of position (the direction of the vector changes with position).
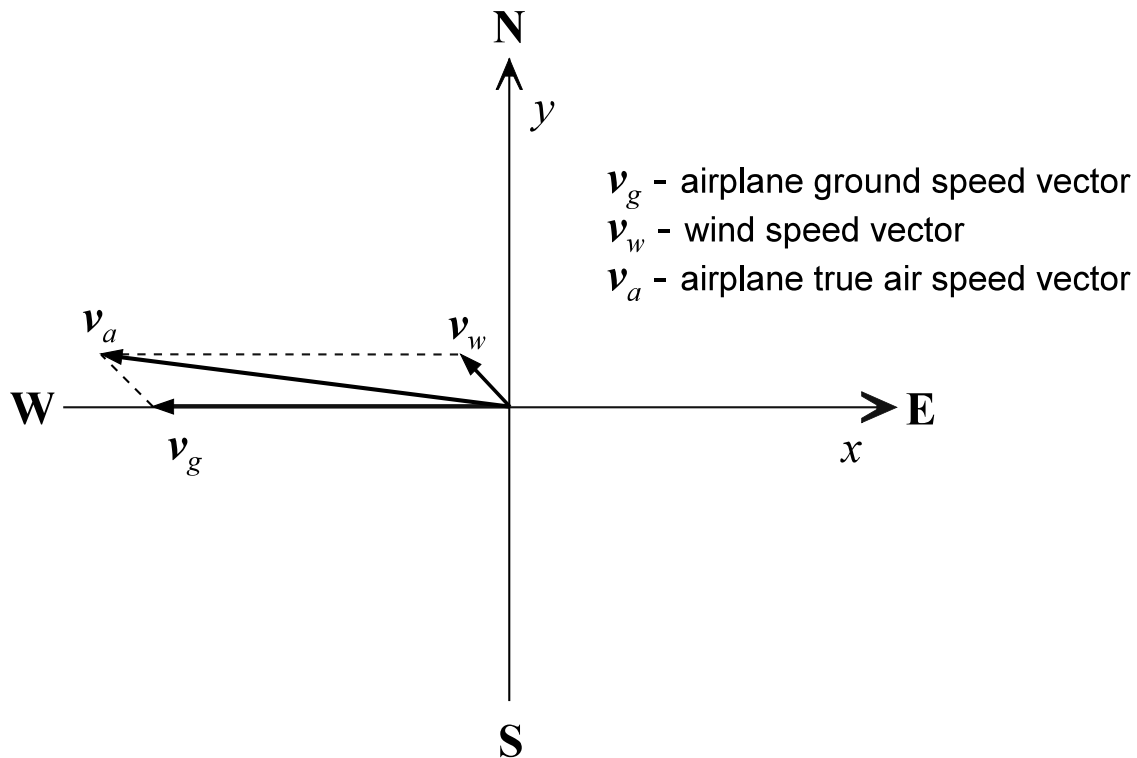
$$\hat{a}_E = \frac{E}{|E|} = \frac{(x + y)\hat{x} + 3\hat{y} + z^2 \hat{z}}{\sqrt{(x + y)^2 + 9 + z^4}} \qquad \left(\begin{array}{c} \text{unit vector as a} \\ \text{function of position} \end{array} \right)$$

At the point $(1,1,1)$ $[x = 1, y = 1, z = 1]$, the unit vector is

$$\left| \hat{a}_E = \frac{1}{\sqrt{14}} (2\hat{x} + 3\hat{y} + \hat{z}) \right|$$

Example (Vector addition)

An airplane with a ground speed of 350 km/hr heading due west flies in a wind blowing to the northwest at 40 km/hr. Determine the true air speed and heading of the airplane.

$$v_g = 350(-\hat{x}) = -350\hat{x}$$

$$v_w = 40\cos 45^o(-\hat{x}) + 40\sin 45^o(\hat{y}) = -28.3\hat{x} + 28.3\hat{y}$$

$$v_a = v_g + v_w = -350\hat{x} - 28.3\hat{x} + 28.3\hat{y} = -378.3\hat{x} + 28.3\hat{y}$$

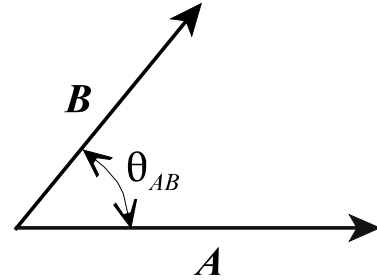$$|v_a| = \sqrt{(378.3)^2 + (28.3)^2} = 379.4 \text{ km/hr}$$

$$\theta = \tan^{-1}\frac{28.3}{378.3} = 4.28^o \text{ north of west}$$

Dot Product
(Scalar Product)

The *dot product* of two vectors A and B (denoted by $A \cdot B$) is defined as the product of the vector magnitudes and the cosine of the smaller angle between them.

$$A \cdot B = |A||B| \cos\theta_{AB} = AB\cos\theta_{AB}$$

$$A \cdot B = B \cdot A \qquad (\text{commutative law})$$

The dot product is commonly used to determine the component of a vector in a particular direction. The dot product of a vector with a unit vector yields the component of the vector in the direction of the unit vector. Given two vectors A and B with corresponding unit vectors a_A and a_B, the component of A in the direction of B (the *projection* of A onto B) is found evaluating the dot product of A with a_B. Similarly, the component of B in the direction of A (the *projection* of B onto A) is found evaluating the dot product of B with a_A.

$$A \cdot \hat{a}_B = |A||\hat{a}_B| \cos\theta_{AB} = A\cos\theta_{AB}$$

$$B \cdot \hat{a}_A = |B||\hat{a}_A| \cos\theta_{AB} = B\cos\theta_{AB}$$
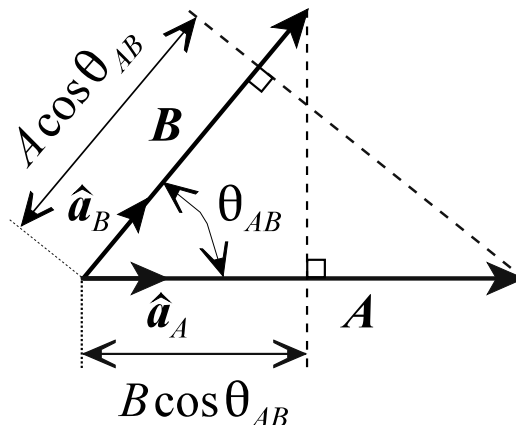
The dot product can be expressed independent of angles through the use of component vectors in an orthogonal coordinate system.

$$A = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$B = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$A \cdot B = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \cdot (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$= A_x B_x \hat{x} \cdot \hat{x} + A_x B_y \hat{x} \cdot \hat{y} + A_x B_z \hat{x} \cdot \hat{z}$$

$$+ A_y B_x \hat{y} \cdot \hat{x} + A_y B_y \hat{y} \cdot \hat{y} + A_y B_z \hat{y} \cdot \hat{z}$$

$$+ A_z B_x \hat{z} \cdot \hat{x} + A_z B_y \hat{z} \cdot \hat{y} + A_z B_z \hat{z} \cdot \hat{z}$$

The dot product of like unit vectors yields one ($\theta_{AB} = 0°$) while the dot product of unlike unit vectors ($\theta_{AB} = 90°$) yields zero. The dot product results are

$$\hat{x} \cdot \hat{x} = 1 \qquad \hat{x} \cdot \hat{y} = 0 \qquad \hat{x} \cdot \hat{z} = 0$$

$$\hat{y} \cdot \hat{x} = 0 \qquad \hat{y} \cdot \hat{y} = 1 \qquad \hat{y} \cdot \hat{z} = 0$$

$$\hat{z} \cdot \hat{x} = 0 \qquad \hat{z} \cdot \hat{y} = 0 \qquad \hat{z} \cdot \hat{z} = 1$$

The resulting dot product expression is
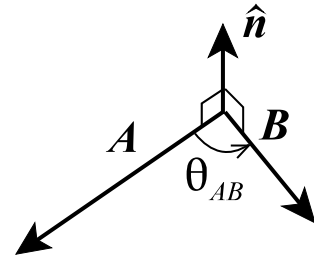
$$\left| \; A \cdot B = A_x B_x + A_y B_y + A_z B_z \; \right|$$

Cross Product
(Vector Product)

The *cross product* of two vectors A and B (denoted by $A \times B$) is defined as the product of the vector magnitudes and the sine of the smaller angle between them with a vector direction defined by the *right hand rule*.

$$A \times B = |A||B| \sin\theta_{AB}\,\hat{n} = AB\sin\theta_{AB}\,\hat{n}$$

$$A \times B = -B \times A \qquad \text{(not commutative)}$$

Note: (1) the unit vector $\hat{n}$ is normal to the plane in which A and B lie.

(2) $AB\sin\theta_{AB}$ = area of the parallelogram formed by the vectors A and B.

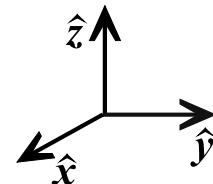Using component vectors, the cross product of A and B may be written as

$$A = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$B = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$A \times B = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= A_xB_x\,\hat{x} \times \hat{x} + A_xB_y\,\hat{x} \times \hat{y} + A_xB_z\,\hat{x} \times \hat{z}$$

$$+ A_yB_x\,\hat{y} \times \hat{x} + A_yB_y\,\hat{y} \times \hat{y} + A_yB_z\,\hat{y} \times \hat{z}$$

$$+ A_zB_x\,\hat{z} \times \hat{x} + A_zB_y\,\hat{z} \times \hat{y} + A_zB_z\,\hat{z} \times \hat{z}$$

The cross product of like unit vectors yields zero ($\theta_{AB} = 0°$) while the cross product of unlike unit vectors ($\theta_{AB} = 90°$) yields another unit vector which is determined according to the right hand rule. The cross products results are

$$\hat{x} \times \hat{x} = 0 \qquad \hat{x} \times \hat{y} = \hat{z} \qquad \hat{x} \times \hat{z} = -\hat{y}$$

$$\hat{y} \times \hat{x} = -\hat{z} \qquad \hat{y} \times \hat{y} = 0 \qquad \hat{y} \times \hat{z} = \hat{x}$$

$$\hat{z} \times \hat{x} = \hat{y} \qquad \hat{z} \times \hat{y} = -\hat{x} \qquad \hat{z} \times \hat{z} = 0$$

The resulting cross product expression is

$$A \times B = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

This cross product result can also be written compactly in the form of a determinant as

$$A \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example (Dot product / Cross product)

Given $E = 3\hat{y} + 4\hat{z}$ and $F = 4\hat{x} - 10\hat{y} + 5\hat{z}$, determine
  (a.) the vector component of E in the direction of F.
  (b.) a unit vector perpendicular to both E and F.

(a.) To find the vector component of E in the direction of F, we must dot the vector E with the unit vector in the direction of F.

$$\hat{a}_F = \frac{F}{|F|} = \frac{4\hat{x} - 10\hat{y} + 5\hat{z}}{\sqrt{4^2 + 10^2 + 5^2}} = \frac{1}{\sqrt{141}}(4\hat{x} - 10\hat{y} + 5\hat{z})$$

The dot product of E and a_F is

$$E \cdot \hat{a}_F = (3\hat{y} + 4\hat{z}) \cdot \frac{1}{\sqrt{141}}(4\hat{x} - 10\hat{y} + 5\hat{z})$$

$$= \frac{1}{\sqrt{141}}[(3)(-10) + (4)(5)] = -\frac{10}{\sqrt{141}}$$

(Scalar component of E along F)

The vector component of E along F is

$$(E \cdot \hat{a}_F)\hat{a}_F = -\frac{10}{141}(4\hat{x} - 10\hat{y} + 5\hat{z})$$

(b.) To find a unit vector normal to both E and F, we use the cross product. The result of the cross product is a vector which is normal to both E and F.

$$E \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55\hat{x} + 16\hat{y} - 12\hat{z})$$

We then divide this vector by its magnitude to find the unit vector.

$$\hat{n} = \frac{E \times F}{|E \times F|} = \frac{55\hat{x} + 16\hat{y} - 12\hat{z}}{\sqrt{55^2 + 16^2 + 12^2}} = \frac{1}{\sqrt{3425}}(55\hat{x} + 16\hat{y} - 12\hat{z})$$

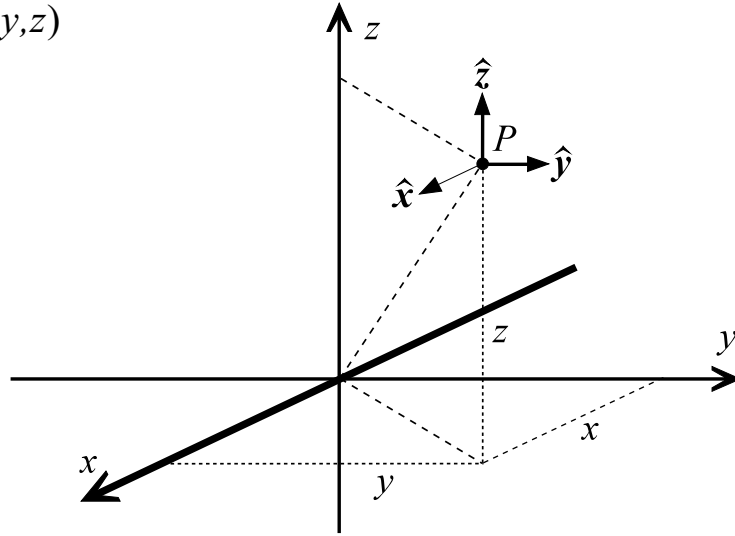The negative of this unit vector is also normal to both E and F.

Coordinate and Unit Vector Definitions

Rectangular Coordinates (x,y,z)

$$(-\infty < x < \infty)$$
$$(-\infty < y < \infty)$$
$$(-\infty < z < \infty)$$

Cylindrical Coordinates (r,ϕ,z)

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\phi$$
$$\phi = \tan^{-1}(y/x) \qquad y = r\sin\phi$$
$$z = z \qquad\qquad z = z$$

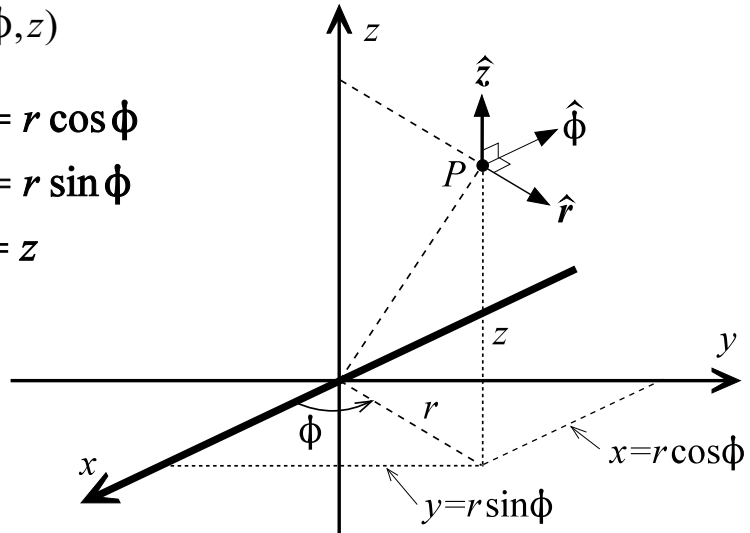$$(0 \le r < \infty)$$
$$(0 \le \phi < 2\pi)$$
$$(-\infty < z < \infty)$$

Spherical Coordinates (R,θ,ϕ)

$$R = \sqrt{x^2 + y^2 + z^2} \qquad x = R\sin\theta\cos\phi$$
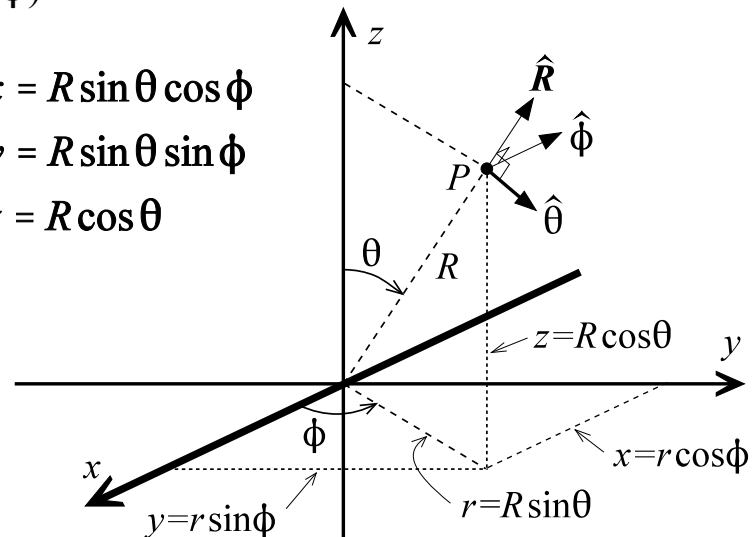$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad y = R\sin\theta\sin\phi$$
$$\phi = \tan^{-1}(y/x) \qquad\qquad z = R\cos\theta$$

$$(0 \le R < \infty)$$
$$(0 \le \theta \le \pi)$$
$$(0 \le \phi < 2\pi)$$

Vector Definitions and Coordinate Transformations

Vector Definitions

$\qquad$ Rectangular $\qquad A = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} = (A_x, A_y, A_z)$

$\qquad$ Cylindrical $\qquad A = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{z} = (A_r, A_\phi, A_z)$

$\qquad$ Spherical $\qquad A = A_R \hat{R} + A_\theta \hat{\theta} + A_\phi \hat{\phi} = (A_R, A_\theta, A_\phi)$

Vector Magnitudes

$$A \cdot A = |A||A| \cos 0^o = |A|^2 \qquad \Rightarrow \qquad |A| = \sqrt{A \cdot A}$$

$\qquad$ Rectangular $\qquad |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$\qquad$ Cylindrical $\qquad |A| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$

$\qquad$ Spherical $\qquad |A| = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$

Rectangular to Cylindrical Coordinate Transformation

$$(A_x, A_y, A_z) \quad \Rightarrow \quad (A_r, A_\phi, A_z)$$

The transformation of rectangular to cylindrical coordinates requires that we find the components of the rectangular coordinate vector A in the direction of the cylindrical coordinate unit vectors (using the dot product). The required dot products are

$$A_r = A \cdot \hat{r} = A_x \hat{x} \cdot \hat{r} + A_y \hat{y} \cdot \hat{r} + A_z \hat{z} \cdot \hat{r} = A_x \hat{x} \cdot \hat{r} + A_y \hat{y} \cdot \hat{r}$$

$$A_\phi = A \cdot \hat{\phi} = A_x \hat{x} \cdot \hat{\phi} + A_y \hat{y} \cdot \hat{\phi} + A_z \hat{z} \cdot \hat{\phi} = A_x \hat{x} \cdot \hat{\phi} + A_y \hat{y} \cdot \hat{\phi}$$

$$A_z = A \cdot \hat{z} = A_x \hat{x} \cdot \hat{z} + A_y \hat{y} \cdot \hat{z} + A_z \hat{z} \cdot \hat{z} = A_z$$
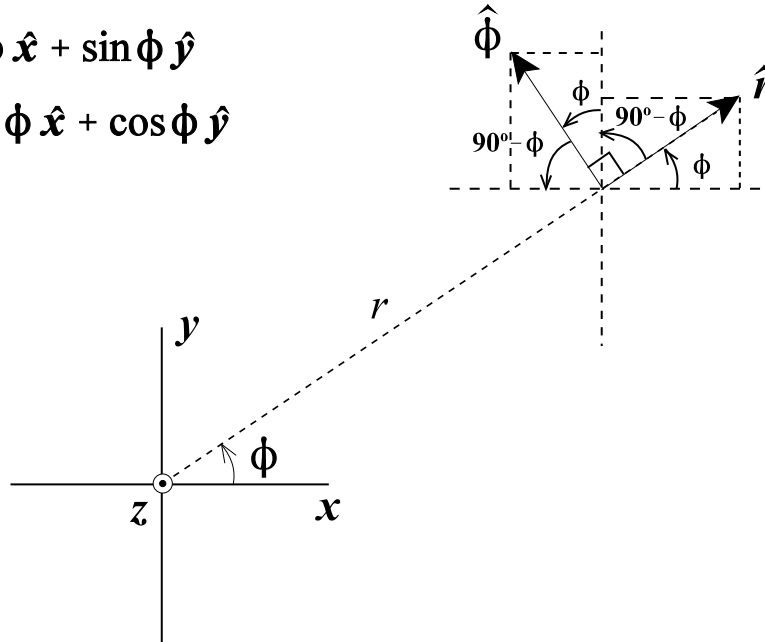
where the $\hat{z}$ unit vector is identical in both orthogonal coordinate systems such that

$$\hat{z} \cdot \hat{r} = 0 \qquad \hat{z} \cdot \hat{\phi} = 0$$

$$\hat{x} \cdot \hat{z} = 0 \qquad \hat{y} \cdot \hat{z} = 0 \qquad \hat{z} \cdot \hat{z} = 1$$

The four remaining unit vector dot products are determined according to the geometry relationships between the two coordinate systems.

$$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{x} \cdot \hat{r} = \hat{x} \cdot (\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = \cos\phi$$

$$\hat{y} \cdot \hat{r} = \hat{y} \cdot (\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) = \sin\phi$$

$$\hat{x} \cdot \hat{\phi} = \hat{x} \cdot (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = -\sin\phi$$

$$\hat{y} \cdot \hat{\phi} = \hat{y} \cdot (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \cos\phi$$

The resulting cylindrical coordinate vector is

$$A = A_r\,\hat{r} + A_\phi\,\hat{\phi} + A_z\,\hat{z}$$

$$= (A_x\cos\phi + A_y\sin\phi)\hat{r} + (A_y\cos\phi - A_x\sin\phi)\hat{\phi} + A_z\,\hat{z}$$

In matrix form, the rectangular to cylindrical transformation is

$$
\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}
$$

Cylindrical to Rectangular Coordinate Transformation

$$
(A_r, A_\phi, A_z) \Rightarrow (A_x, A_y, A_z)
$$

The transformation from cylindrical to rectangular coordinates can be determined as the inverse of the rectangular to cylindrical transformation.

$$
\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}
$$

The cylindrical coordinate variables in the transformation matrix must be expressed in terms of rectangular coordinates.

$$
\cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}
$$

$$
\sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}
$$

The resulting transformation is

$$
\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & -\dfrac{y}{\sqrt{x^2+y^2}} & 0 \\ \dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}
$$

The cylindrical to rectangular transformation can be written as

$$
\begin{aligned}
\boldsymbol{A} &= A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}} \\
&= (A_r \cos\phi - A_\phi \sin\phi)\hat{\boldsymbol{x}} + (A_r \sin\phi + A_\phi \cos\phi)\hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}} \\
&= \left(A_r \frac{x}{\sqrt{x^2+y^2}} - A_\phi \frac{y}{\sqrt{x^2+y^2}} \right)\hat{\boldsymbol{x}} \\
&\quad + \left(A_r \frac{y}{\sqrt{x^2+y^2}} + A_\phi \frac{x}{\sqrt{x^2+y^2}} \right)\hat{\boldsymbol{y}} \\
&\quad + A_z \hat{\boldsymbol{z}}
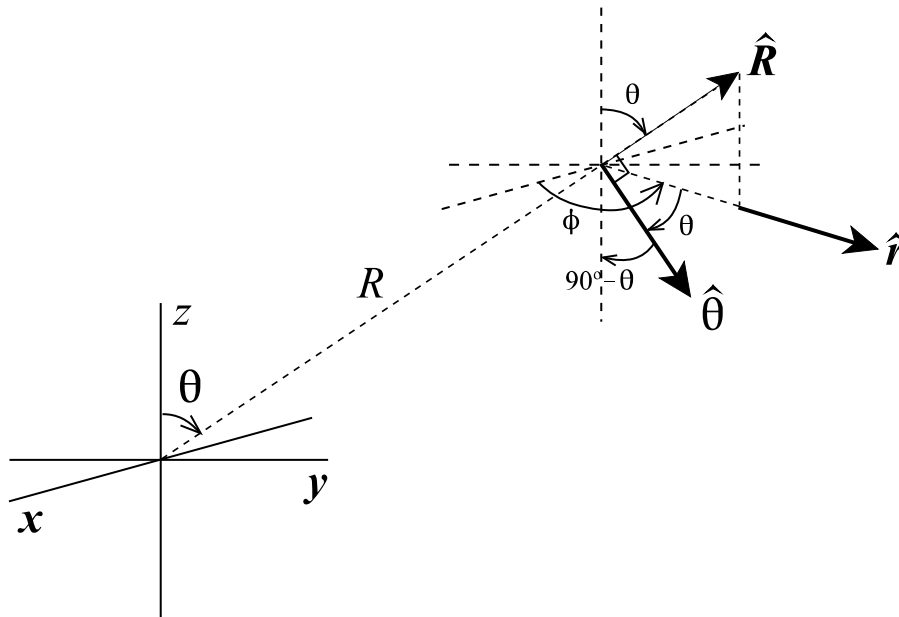\end{aligned}
$$

Rectangular to Spherical Coordinate Transformation

$$
(A_x, A_y, A_z) \ \Rightarrow \ (A_R, A_\theta, A_\phi)
$$

The dot products necessary to determine the transformation from rectangular coordinates to spherical coordinates are

$$
A_r = \boldsymbol{A} \cdot \hat{\boldsymbol{R}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{R}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{R}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}
$$

$$
A_\theta = \boldsymbol{A} \cdot \hat{\boldsymbol{\theta}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\theta}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\theta}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}}
$$

$$
A_\phi = \boldsymbol{A} \cdot \hat{\boldsymbol{\phi}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\phi}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\phi}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}}
$$

The geometry relationships between the rectangular and spherical unit vectors are illustrated below.

$$\hat{\boldsymbol{R}} = \sin\theta\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{z}}$$
$$= \sin\theta\,[\,\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}\,] + \cos\theta\,\hat{\boldsymbol{z}}$$
$$= \sin\theta\,\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\,\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{r}} - \cos(90^o - \theta)\,\hat{\boldsymbol{z}}$$
$$= \cos\theta\,[\,\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}\,] - \sin\theta\,\hat{\boldsymbol{z}}$$
$$= \cos\theta\,\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\,\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$$

The dot products are then

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{R}} = \sin\theta\cos\phi \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{R}} = \sin\theta\sin\phi \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}} = \cos\theta$$

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\theta}} = \cos\theta\sin\phi \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\theta}} = -\sin\theta$$

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\phi}} = -\sin\phi \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\phi}} = \cos\phi \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}} = 0$$

and the rectangular to spherical transformation may be written as

$$
\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}
$$

Spherical to Rectangular Coordinate Transformation

$$(A_R, A_\theta, A_\phi) \;\Rightarrow\; (A_x, A_y, A_z)$$

The spherical to rectangular coordinate transformation is the inverse of the rectangular to spherical coordinate transformation so that

$$
\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\sin\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}
$$

The spherical coordinate variables in terms of the rectangular coordinate variables are

$$
\sin\theta = \frac{r}{R} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \cos\theta = \frac{z}{R} = \frac{z}{\sqrt{x^2+y^2+z^2}}
$$

$$
\sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}
$$

The complete spherical to rectangular coordinate transformation is

$$
\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} =
\begin{bmatrix}
\dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{xz}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}} & \dfrac{-y}{\sqrt{x^2+y^2}} \\[4ex]
\dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{yz}{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2+z^2}} & \dfrac{x}{\sqrt{x^2+y^2}} \\[4ex]
\dfrac{z}{\sqrt{x^2+y^2+z^2}} & -\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0
\end{bmatrix}
\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}
$$

Coordinate Transformation Procedure

(1) Transform the component scalars into the new coordinate system.

(2) Insert the component scalars into the coordinate transformation matrix and evaluate.

Steps (1) and (2) can be performed in either order.

<u>Example</u> (Coordinate Transformations)

Given the rectangular coordinate vector

$$A = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \hat{x} - \frac{yz}{\sqrt{x^2 + y^2 + z^2}} \hat{z}$$

(a.) transform the vector A into cylindrical and spherical coordinates.

(b.) transform the rectangular coordinate point P (1,3,5) into cylindrical and spherical coordinates.

(c.) evaluate the vector A at P in rectangular, cylindrical and spherical coordinates.

(a.)
$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

$$A_x = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} = \frac{r}{\sqrt{r^2 + z^2}}$$

$$A_z = -\frac{yz}{\sqrt{x^2 + y^2 + z^2}} = -\frac{zr\sin\phi}{\sqrt{r^2 + z^2}}$$

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{r}{\sqrt{r^2 + z^2}} \\ 0 \\ -\dfrac{zr\sin\phi}{\sqrt{r^2 + z^2}} \end{bmatrix}$$

$$A_r = \frac{r\cos\phi}{\sqrt{r^2+z^2}} \qquad A_\phi = -\frac{r\sin\phi}{\sqrt{r^2+z^2}} = \qquad A_z = -\frac{zr\sin\phi}{\sqrt{r^2+z^2}}$$

$$\left| \quad \mathbf{A} = \frac{r}{\sqrt{r^2+z^2}} \left(\cos\phi\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}} - z\sin\phi\,\hat{\mathbf{z}} \right) \quad \right|$$

$$x = R\sin\theta\cos\phi \qquad A_x = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \frac{R\sin\theta}{R} = \sin\theta$$

$$y = R\sin\theta\sin\phi$$

$$z = R\cos\theta \qquad\qquad A_z = -\frac{yz}{\sqrt{x^2+y^2+z^2}} = -\frac{R^2\sin\theta\cos\theta\sin\phi}{R}$$

$$= -R\sin\theta\cos\theta\sin\phi$$

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \sin\theta \\ 0 \\ -R\sin\theta\cos\theta\sin\phi \end{bmatrix}$$

$$A_R = \sin^2\theta\cos\phi - R\sin\theta\cos^2\theta\sin\phi$$

$$A_\theta = \sin\theta\cos\theta\cos\phi + R\sin^2\theta\cos\theta\sin\phi$$

$$A_\phi = -\sin\theta\sin\phi$$

$$\left| \quad \begin{aligned} \mathbf{A} &= \sin\theta\,(\,\sin\theta\cos\phi - R\cos^2\theta\sin\phi\,)\,\hat{\mathbf{R}} \\ &\quad + \sin\theta\cos\theta\,(\,\cos\phi + R\sin\theta\sin\phi\,)\,\hat{\boldsymbol{\theta}} \\ &\quad - \sin\theta\sin\phi\,\hat{\boldsymbol{\phi}} \end{aligned} \quad \right|$$

(b.) $P\,(1,3,5) \quad \Rightarrow \quad x = 1, y = 3, z = 5$

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 3^2} = \sqrt{10} = 3.16$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/1) = 71.6^o$$

$$z = z = 5$$

$$R = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35} = 5.92$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{1^2 + 3^2}}{5}\right) = 32.3^o$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/1) = 71.6^o$$

$$\boxed{P\,(1, 3, 5) \quad \Rightarrow \quad P\,(3.16, 71.6^o, 5) \quad \Rightarrow \quad P\,(5.92, 32.3^o, 71.6^o)}$$

(c.)
$$A(1,3,5) = \frac{\sqrt{1^2 + 3^2}}{\sqrt{1^2 + 3^2 + 5^2}}\,\hat{x} - \frac{(3)(5)}{\sqrt{1^2 + 3^2 + 5^2}}\,\hat{z} = \boxed{0.535\,\hat{x} - 2.54\,\hat{z}}$$

$$A(3.16, 71.6^o, 5) = \frac{3.16}{\sqrt{3.16^2 + 5^2}}\,(\cos 71.6^o\,\hat{r} - \sin 71.6^o\,\hat{\phi}$$

$$- 5\sin 71.6^o\,\hat{z})$$

$$= \boxed{0.169\,\hat{r} - 0.507\,\hat{\phi} - 2.53\,\hat{z}}$$

$$A(5.92, 32.3^o, 71.6^o) =$$

$$\sin 32.3^o\,(\sin 32.3^o \cos 71.6^o - 5.92\cos^2 32.3^o \sin 71.6^o)\,\hat{R}$$

$$+ \sin 32.3^o \cos 32.3^o\,(\cos 71.6^o + 5.92\sin 32.3^o \sin 71.6^o)\,\hat{\theta}$$

$$- \sin 32.3^o \sin 71.6^o\,\hat{\phi}$$

$$= \boxed{-2.05\,\hat{R} + 1.50\,\hat{\theta} + 0.507\,\hat{\phi}}$$

Separation Distances

Given a vector $\boldsymbol{R}_1$ locating the point P_1 and a vector $\boldsymbol{R}_2$ locating the point P_2, the distance d between the points is found by determining the magnitude of the vector pointing from P_1 to P_2, or vice versa.
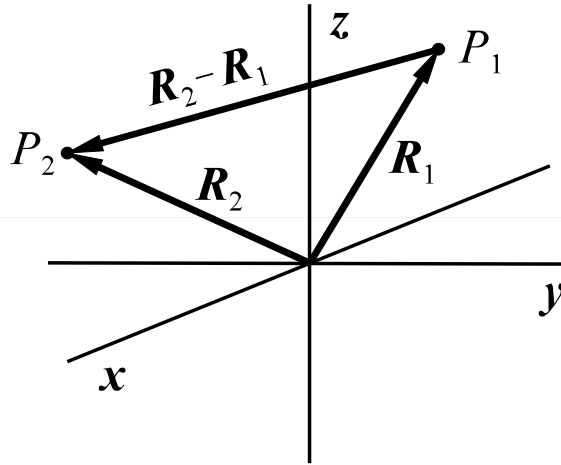
$$d = |\boldsymbol{R}_2 - \boldsymbol{R}_1| = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$$

Rectangular

$$P_1 = (x_1, y_1, z_1) \qquad P_2 = (x_2, y_2, z_2)$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Cylindrical

$$P_1 = (r_1, \phi_1, z_1) \qquad P_2 = (r_2, \phi_2, z_2)$$

$$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

Spherical

$$P_1 = (R_1, \theta_1, \phi_1) \qquad P_2 = (R_2, \theta_2, \phi_2)$$

$$d = \sqrt{R_2^2 + R_1^2 - 2 R_1 R_2 \cos\theta_2 \cos\theta_1 - 2 R_1 R_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)}$$

Constant Coordinate Surfaces

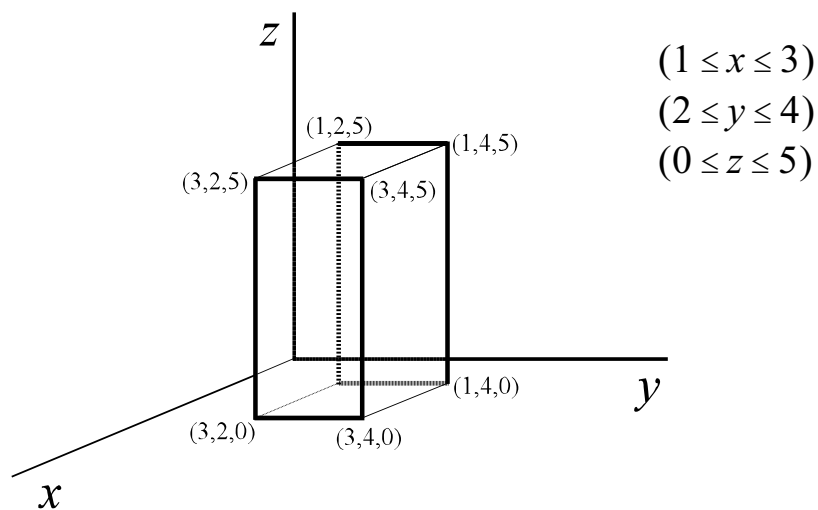## Rectangular Coordinates
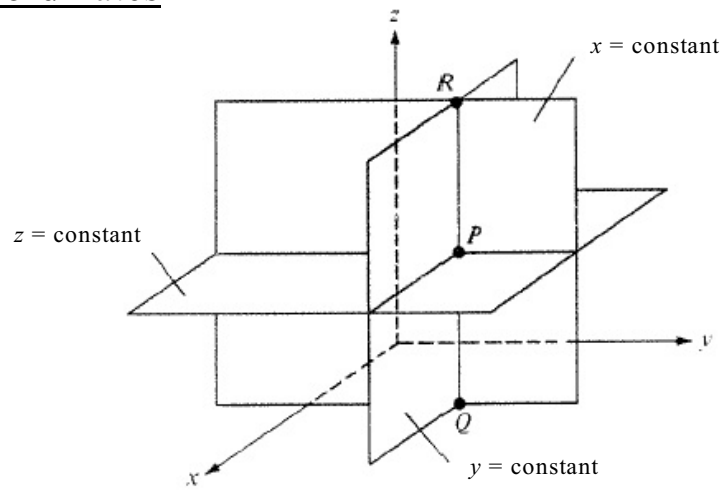

Cylindrical Coordinates
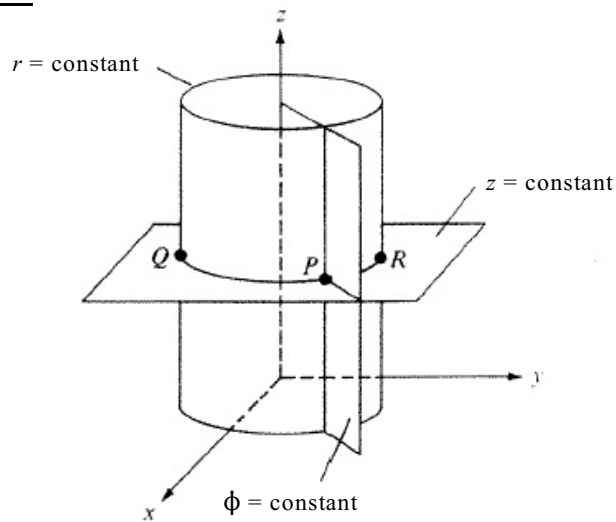
Spherical Coordinates
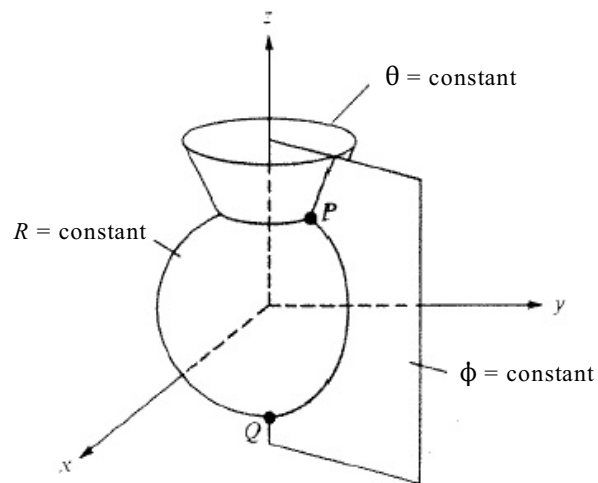
Volumes, Surfaces and Lines in Rectangular, Cylindrical and Spherical Coordinates

We may define particular volumes, surfaces and lines in rectangular, cylindrical and spherical coordinates by specifying ranges on the coordinate variables.
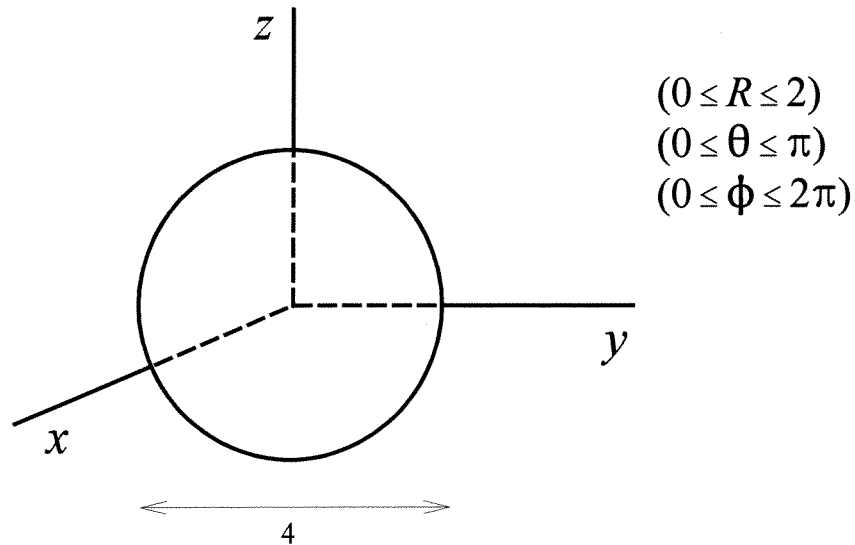
Rectangular volume (2×2×5 box)

$$(1 \leq x \leq 3)$$
$$(2 \leq y \leq 4)$$
$$(0 \leq z \leq 5)$$

Cylindrical volume (cylinder of length = 5, diameter = 2)

$$(0 \leq r \leq 1)$$
$$(0 \leq \phi \leq 2\pi)$$
$$(0 \leq z \leq 5)$$
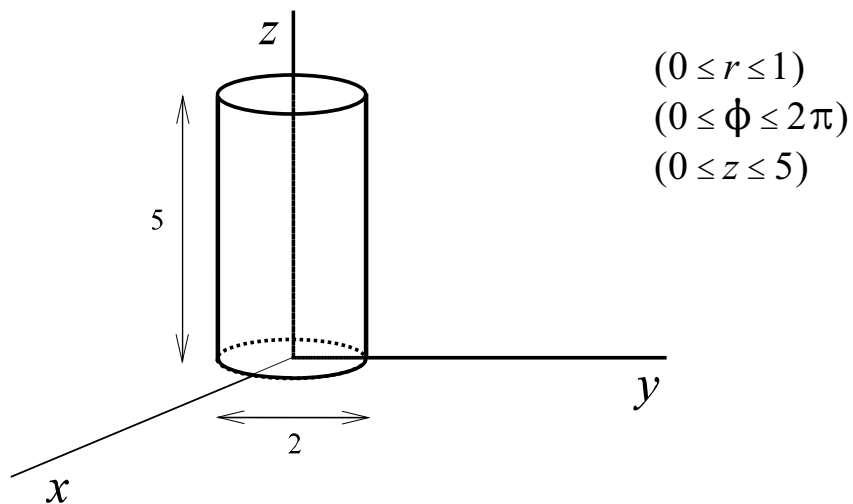
Spherical volume (sphere of diameter = 4)

$(0 \le R \le 2)$
$(0 \le \theta \le \pi)$
$(0 \le \phi \le 2\pi)$

Specific surfaces and lines can be defined in a given coordinate system according to which coordinate variable(s) is(are) held constant. A surface results when one of the coordinate variables is held constant while a line results when two of the coordinate variables are held constant.
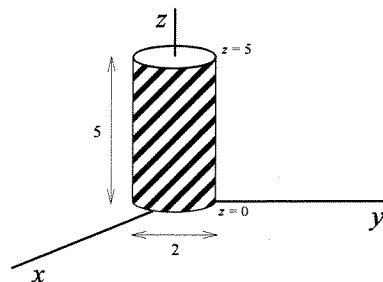
Surface on the Rectangular volume
(front face of the box)

$x = 3$ $(x - \text{constant})$
$(2 \le y \le 4)$
$(0 \le z \le 5)$
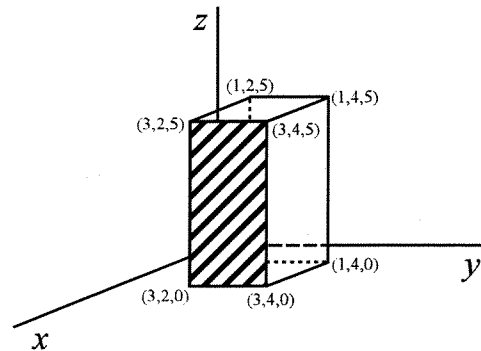
Surface on the Cylindrical volume
(cylinder surface)

$r = 1$ $(r - \text{constant})$
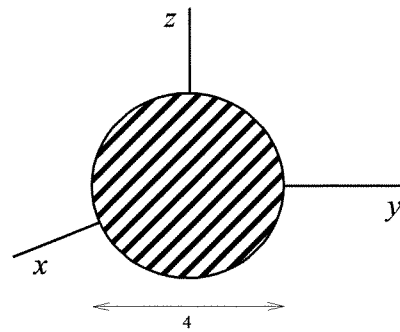$(0 \le \phi \le 2\pi)$
$(0 \le z \le 5)$

Surface on the Spherical volume
(outer surface of the sphere)

$R = 2$ $\qquad$ (r – constant)

$(0 \le \theta \le \pi)$

$(0 \le \phi \le 2\pi)$

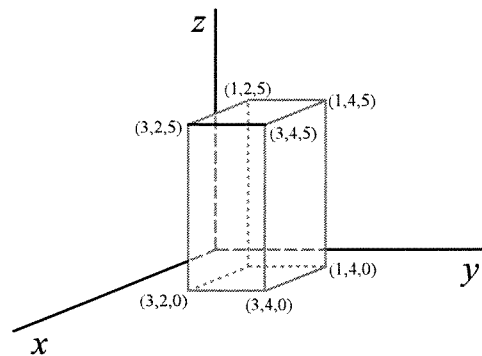## Line on the Rectangular volume
(upper edge of the front face)

$x = 3$ $\qquad$ (x – constant)

$(2 \le y \le 4)$

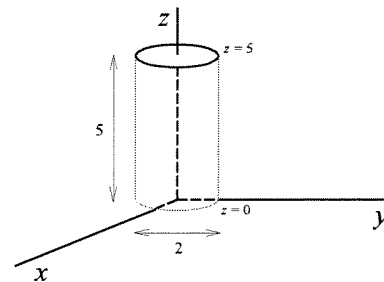$z = 5$ $\qquad$ (z – constant)

Line on the Cylindrical volume
(outer edge of the upper surface)

$r = 1$ $\qquad$ (r – constant)

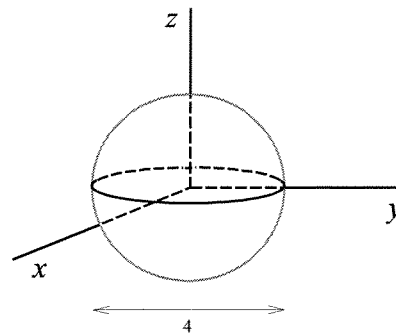$(0 \le \phi \le 2\pi)$

$z = 5$ $\qquad$ (z – constant)

Line on the Spherical volume
(equator of the sphere)

$R = 2$ $\qquad$ (R – constant)
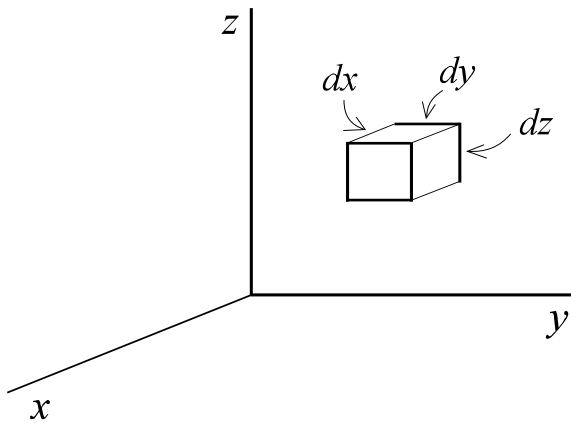
$\theta = \pi/2$ $\qquad$ (θ – constant)

$(0 \le \phi \le 2\pi)$

Differential Lengths, Surfaces and Volumes

When integrating along lines, over surfaces, or throughout volumes, the ranges of the respective variables define the limits of the respective integrations. In order to evaluate these integrals, we must properly define the differential elements of length, surface and volume in the coordinate system of interest. The definition of the proper differential elements of length (*dl* for line integrals) and area (*ds* for surface integrals) can be determined directly from the definition of the differential volume (*dv* for volume integrals) in a particular coordinate system.

Rectangular Coordinates

$$dv = (dx)(dy)(dz)$$

x-constant	$ds = dy\,dz$
y-constant	$ds = dx\,dz$
z-constant	$ds = dx\,dy$
x,y-constant	$dl = dz$
x,z-constant	$dl = dy$
y,z-constant	$dl = dx$

Cylindrical Coordinates
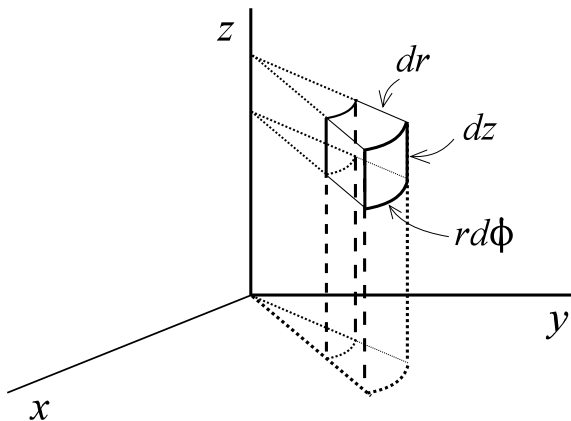
$$dv = (dr)(r\,d\phi)(dz)$$
$$= r\,dr\,d\phi\,dz$$

r-constant	$ds = r_o\,d\phi\,dz$
ϕ-constant	$ds = dr\,dz$
z-constant	$ds = r\,dr\,d\phi$
r,ϕ-constant	$dl = dz$
r,z-constant	$dl = r_o\,d\phi$
ϕ,z-constant	$dl = dr$

Spherical Coordinates
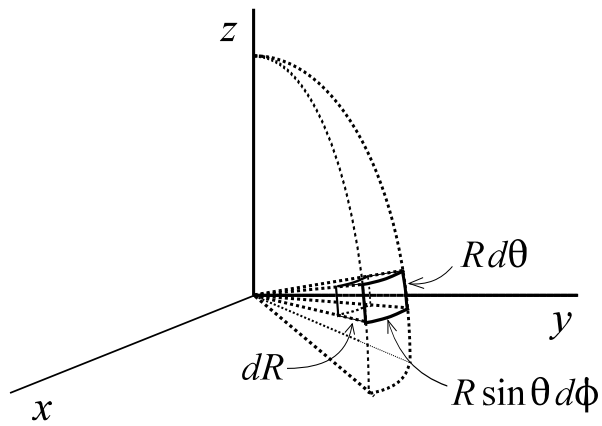
$$dv = (dR)(R\,d\theta)(R\sin\theta\,d\phi)$$

$$= R^2 \sin\theta\, dR\, d\theta\, d\phi$$

R-constant $\qquad ds = R_o^2 \sin\theta\, d\theta\, d\phi$

θ-constant $\qquad ds = R\sin\theta_o\, dR\, d\phi$

ϕ-constant $\qquad ds = R\, dR\, d\theta$

R,θ-constant $\qquad dl = R_o \sin\theta_o\, d\phi$

R,ϕ-constant $\qquad dl = R_o\, d\theta$

θ,ϕ-constant $\qquad dl = dR$

<u>Example</u> (Line/surface/volume integration)

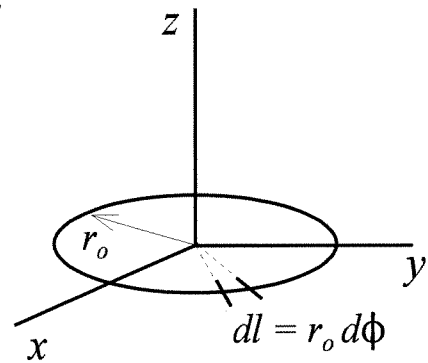Using the appropriate differential elements, show that
- (a.)   the circumference of a circle of radius r_o is $2\pi r_o$.
- (b.)   the surface area of a sphere of radius R_o is $4\pi R_o^2$.
- (c.)   the volume of a sphere of radius R_o is $(4/3)\pi R_o^3$.

(a.)

$$L = \int dl = \int_0^{2\pi} r_o\, d\phi = r_o \int_0^{2\pi} d\phi = r_o [\phi]\Big|_0^{2\pi}$$

$$\boxed{L = 2\pi r_o}$$

$dl = r_o\, d\phi$

(b.)

$$S = \iint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} R_o^2 \sin\theta\, d\theta\, d\phi$$
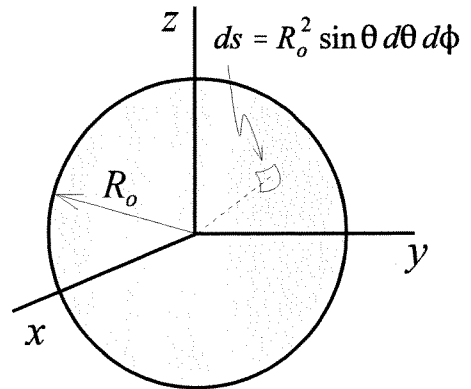
$$= R_o^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta\, d\theta\, d\phi$$

$$= R_o^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$
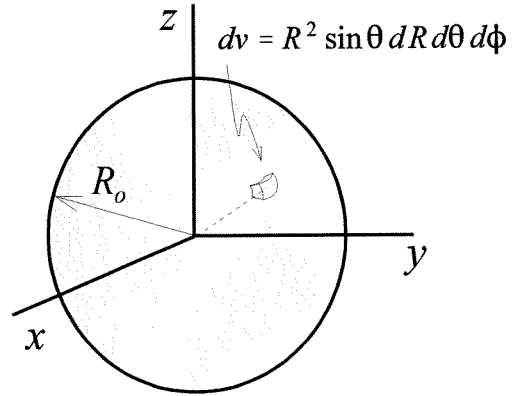
$$= R_o^2 [-\cos\theta]\Big|_0^{\pi} [\phi]\Big|_0^{2\pi}$$

$$= R_o^2 (2)(2\pi)$$

$$\boxed{S = 4\pi R_o^2}$$

$ds = R_o^2 \sin\theta\, d\theta\, d\phi$

(c.)

$$V = \iiint dv$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{R_o} R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \int_{0}^{R_o} R^2 \, dR \int_{0}^{\pi} \sin\theta \, d\theta \int_{0}^{2\pi} d\phi$$

$$= \left[\frac{R^3}{3} \right] \Bigg|_{0}^{R_o} \left[-\cos\theta \right] \Big|_{0}^{\pi} \left[\phi \right] \Big|_{0}^{2\pi}$$

$$= \left(\frac{R_o^3}{3} \right) (2)(2\pi)$$

$$\boxed{V = \frac{4}{3} \pi R_o^3}$$

z

$dv = R^2 \sin\theta \, dR \, d\theta \, d\phi$

R_o

y

x

Example (Surface/volume integration in spherical coordinates)

A three-dimensional solid is described in spherical coordinates according to

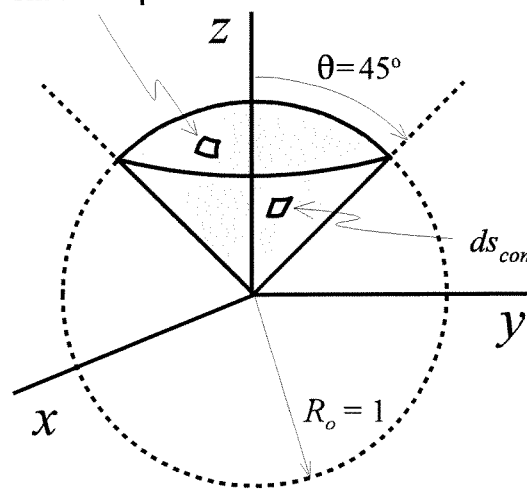$$(0 \le R \le 1) \qquad (0 \le \theta \le \pi/4) \qquad (0 \le \phi \le 2\pi)$$

(a.) Sketch the solid.
(b.) Determine the volume of the solid.
(c.) Determine the surface area of the solid

(a.)

spherical coordinate differential volume
$$dv = (dR)(R\,d\theta)(R\sin\theta\,d\phi)$$

$ds_{top} = R_o^2 \sin\theta\,d\theta\,d\phi = \sin\theta\,d\theta\,d\phi$
$(R_o = 1)$

$\theta = 45°$

z

$ds_{cone} = R\sin\theta_o\,dR\,d\phi = \dfrac{1}{\sqrt{2}}R\,dR\,d\phi$
$(\theta_o = 45°)$

y

$R_o = 1$

x

(b.)

$$V = \iiint dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{R=0}^{1} R^2 \sin\theta\,dR\,d\theta\,d\phi$$

$$= \int_0^1 R^2\,dR \int_{\theta=0}^{\pi/4} \sin\theta\,d\theta \int_{\phi=0}^{2\pi} d\phi = \left[\frac{R^3}{3}\right]\Bigg|_0^1 \left[-\cos\theta\right]\Big|_0^{\pi/4} \left[\phi\right]\Big|_0^{2\pi}$$

$$= \left(\frac{1}{3}\right)\left(-\frac{1}{\sqrt{2}}+1\right)(2\pi) \qquad \boxed{V = 0.613 \text{ units}^3}$$

(c.)

$$S = S_{top} + S_{cone}$$

$$= \iint\limits_{top\ surface} ds_{top} + \iint\limits_{cone\ surface} ds_{cone}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \sin\theta\, d\theta\, d\phi + \frac{1}{\sqrt{2}} \int_{\phi=0}^{2\pi} \int_{R=0}^{1} R\, dR\, d\phi$$

$$= \int_{0}^{\pi/4} \sin\theta\, d\theta \int_{0}^{2\pi} d\phi + \frac{1}{\sqrt{2}} \int_{0}^{1} R\, dR \int_{0}^{2\pi} d\phi$$

$$= [-\cos\theta]\Big|_{0}^{\pi/4} [\phi]\Big|_{0}^{2\pi} + \frac{1}{\sqrt{2}} \left[\frac{R^2}{2}\right]\Big|_{0}^{1} [\phi]\Big|_{0}^{2\pi}$$

$$= \left[-\frac{1}{\sqrt{2}} + 1\right](2\pi) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)(2\pi)$$

$$S = 4.06 \ units^2$$