Radiophysics and Quantum Electronics, Vol. 52, No. 12, 2009

ELECTROMAGNETIC FIELD OF A ROTATING MAGNETIC DIPOLE AND ELECTRIC-CHARGE MOTION IN THIS FIELD

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In this paper, we present the results of a study of the electromagnetic field in the near and intermediate zones of a magnetic dipole rotating in free space. Examples of solving the relativistic equations of the charged-particle motion in this field are given. The energy, which can be acquired by the particles during acceleration, is estimated.

1. INTRODUCTION

The field of a rotating magnetic dipole is the classical problem of electrodynamics. A similar problem (oscillator field) is solved in [1]. Nevertheless, research workers dealing with the radiation-theory formalism do not have enough knowledge of the details of the geometric structure of the force lines of a rotating magnetic dipole. Such details, mutual orientation, and the ratio of the electric and magnetic fields are mainly required for studies of pulsars. At present, the studies of the structure of electromagnetic fields of the magnetospheres of these objects are far from completion. In fact, the collisionless motion of charged particles in this field has not been studied. Interpretation of the extensive accumulated empirical results of observations of the radio-pulsar impulses is not satisfactory. The author of a brief review of the state of the art in the theory of pulsars [2] indicates very insignificant progress of the 35-year studies in this field. The pulsar theory is mingled with myths, the number of new ideas increases, but no clear understanding of the main operation principles of pulsars is reached. The mechanism of unipolar induction, which was proposed 40 years ago in [3], has still been governing in the pulsar theory. At the same time, the strength of the rotational electric field induced by the dipole rotation is not allowed for. This field can significantly exceed the field produced by unipolar induction.

Without claiming the discovery of the mechanisms of pulsar functioning, in this paper we solve only two problems: (i) a model of the electromagnetic field of a rotating magnetic dipole is developed under the assumption that its physical dimensions are much smaller than the radiation wavelength and the dipole field is weakly influenced and (ii) 2) the collisionless motion of the charged particles in the electromagnetic field of a rotating dipole is studied.

2. ELECTROMAGNETIC FIELD OF A ROTATING MAGNETIC DIPOLE

To calculate the electromagnetic field of a magnetic rotator, we can use the expression [1] for the retarded vector potential of the dipole with a varying magnetic moment:

$$\mathbf{A} = \frac{[\boldsymbol{\mu}\mathbf{r}]}{r^3} + \frac{[\boldsymbol{\mu}'\mathbf{r}]}{cr^2}.$$

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Tomsk State University, Tomsk, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 52, No. 12, pp. 996–1004, December 2009. Original article submitted November 2, 2009; accepted December 25, 2009.

 0033-8443/09/5212-0900

Here, $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ are the magnetic moment of the dipole and the time derivative of this moment, respectively, and c is the speed of light.

For a dipole rotating with the angular velocity ω, the retarded vector potential $\mathbf{A}$ is determined by the expression

$$\mathbf{A} = \mu\frac{[\mathbf{nr}]}{r^3} + \mu\frac{2\pi}{\lambda r^2}[\mathbf{n}'\mathbf{r}], \tag{1}$$

where μ is the value of the magnetic moment of the dipole ($\boldsymbol{\mu} = \mu\mathbf{n}$) and $\lambda = 2\pi c/\omega$. The rotation axis $\mathbf{n}_z$ is aligned with the z axis, and the unit vectors $\mathbf{n}$ and $\mathbf{n}'$ are orthogonal to this axis. In this case, the components of these vectors in the Cartesian coordinate system are determined by the expressions

$$\mathbf{n} = \{\cos[\omega\,(t - r/c)], \sin[\omega\,(t - r/c)], 0\}, \qquad \mathbf{n}' = \{-\sin[\omega\,(t - r/c)], \cos[\omega\,(t - r/c)], 0\}.$$

If the dipole axis is not orthogonal to the rotation axis, then Eq. (1) is transformed as

$$\mathbf{A} = \mu\cos\alpha\frac{[\mathbf{n}_z\mathbf{r}]}{r^3} + \left(\mu\frac{[\mathbf{nr}]}{r^3} + \mu\frac{\omega}{r^2}[\mathbf{n}'\mathbf{r}]\right)\sin\alpha, \tag{2}$$

where α is the angle between the rotation axis and magnetic moment of the dipole. Therefore, the resulting field is the superposition of two fields such that one field is the field of a constant dipole whose magnetic moment is aligned with the rotation axis, while the second field is the field of a rotating dipole whose magnetic moment is orthogonal to the rotation axis. Only the second term in Eq. (2) is responsible for the dipole radiation. The fields determined only by this term are considered below.

With allowance for the introduced notations, the components of the vector $\mathbf{A}$ in a spherical coordinate system $\{r, \theta, \varphi\}$ can be represented as follows:

$$A_r = 0, \qquad A_\varphi = \frac{\mu\cos\theta}{r^2}\left(2\pi\frac{r}{\lambda}\sin\Phi - \cos\Phi\right),$$
$$A_\theta = \frac{\mu}{r^2}\left(2\pi\frac{r}{\lambda}\cos\Phi + \sin\Phi\right), \qquad \Phi = \tau - 2\pi r/\lambda - \varphi.$$

Here, $\tau = \omega t$ and the factor $\sin\alpha$ is omitted. To allow for this factor, we should replace μ by $\mu\sin\alpha$.

Therefore, using standard electrodynamic expressions, we easily obtain the expressions for the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$

$$\begin{gathered}
B_r = 2\sin\theta\frac{\mu}{r^3}\left(\cos\Phi - 2\pi\frac{r}{\lambda}\sin\Phi\right),\\
B_\theta = \cos\theta\frac{\mu}{r^3}\left(-\cos\Phi + 2\pi\frac{r}{\lambda}\sin\Phi + \left(2\pi\frac{r}{\lambda}\right)^2\cos\Phi\right),\\
B_\varphi = \frac{\mu}{r^3}\left(-\sin\Phi - 2\pi\frac{r}{\lambda}\cos\Phi + \left(2\pi\frac{r}{\lambda}\right)^2\sin\Phi\right),\\
E_r = 0, \qquad E_\theta = -\frac{2\pi\mu}{\lambda r^2}\left(\cos\Phi - 2\pi\frac{r}{\lambda}\sin\Phi\right),\\
E_\varphi = -\frac{2\pi\mu}{\lambda r^2}\cos\theta\left(\sin\Phi + 2\pi\frac{r}{\lambda}\cos\Phi\right).
\end{gathered} \tag{3}$$

In the spherical coordinate system, the components of the Poynting vector averaged over time τ on the interval $[0, 2\pi]$ are determined by the expressions

$$S_r = \frac{\omega^4\mu^2}{8\pi r^2 c^3}(1 + \cos^2\theta), \qquad S_\theta = 0, \qquad S_\varphi = \frac{\omega\mu^2}{4\pi r^5}\left[1 + \left(2\pi\frac{r}{\lambda}\right)^2\right]\sin\theta. \tag{4}$$

The corresponding expression for the radiated power has the form

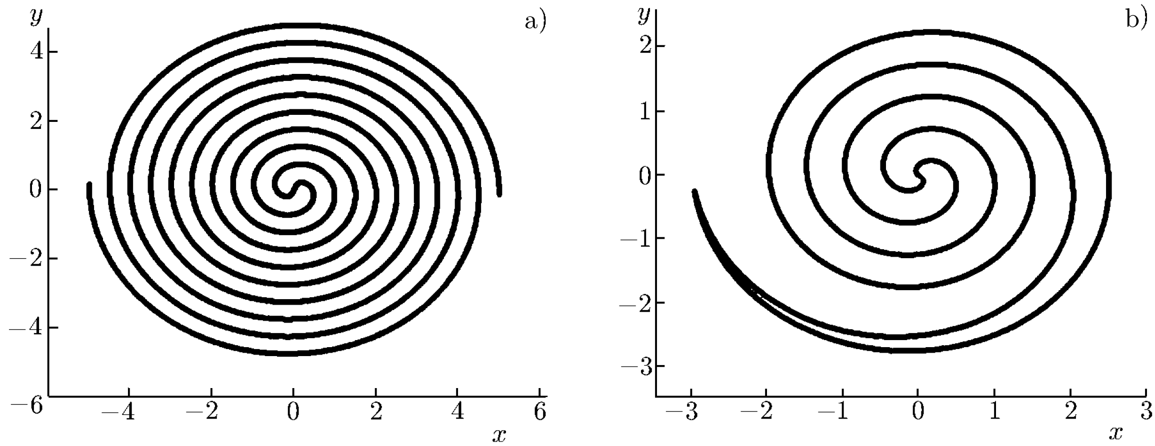

Fig. 1.

$$I = 2\pi \int_0^{\pi} S_r r^2 \sin\theta \, \mathrm{d}\theta = \frac{2}{3} \frac{\omega^4 \mu^2}{c^3}. \qquad (5)$$

It should be noted that Eqs. (3)–(5) are valid at any distance from a dipole, provided that the distance is greater than the physical size of the dipole.

The electromagnetic-field lines are planar and form a set of coplanar circumferences on the sphere surface. The normal to the plane in which these lines are located is determined by the expression

$$\mathbf{k} = 2\pi \mathbf{n} - \frac{\mathbf{n}'}{r}.$$

The magnetic-field lines remain planar only in the equatorial plane $\theta = \pi/2$. In this case, by analogy with the motionless dipole, only the polar force line is open, while all other lines are closed and have return points. Therefore, the magnetic-field lines are not broken during the dipole rotation and are only extended and deformed. Figure 1 shows the examples of such lines, i.e., Fig. 1*a* shows the open polar magnetic-field line and Fig. 1*b* shows the closed equatorial magnetic-field line. Hereafter the Cartesian coordinates x and y in the plane $\theta = \pi/2$ are normalized to λ.

The line shapes were calculated by using the Runge–Kutta method for solving the system of ordinary differential equations

$$\mathrm{d}\mathbf{r}/\mathrm{d}s = \mathbf{B}/|\mathbf{B}|,$$

where $\mathrm{d}s$ is the element of the magnetic-field line.

Although the magnetic-field lines which are not located in the equatorial plane $\theta = \pi/2$ cease to be planar, they are all closed. Figures 2*a* and 2*b* show the examples of such lines (in Fig. 2*b*, the line is drawn to the return point to avoid overloading the figure, and the shape of the line returning to the dipole is the same as that leaving the dipole).

The character of interaction of the electromagnetic radiation with charged particles depends on the vector and scalar product of the fields $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{Z} = [\mathbf{EB}]/B^2, \qquad \cos(\mathbf{E}^\wedge\mathbf{B}) = \mathbf{EB}/(|\mathbf{E}|\,|\mathbf{B}|), \qquad \langle\mathbf{EB}\rangle = \frac{\omega\mu^2}{cr^5}\cos\theta.$$

If $|\mathbf{Z}| < 1$, then the motion includes cyclotron rotation and drift. In this case, the particle energy can vary only if the fields $\mathbf{E}$ and $\mathbf{B}$ are not orthogonal to each other. However, for $|\mathbf{Z}| > 1$, the particle energy will vary in any case.

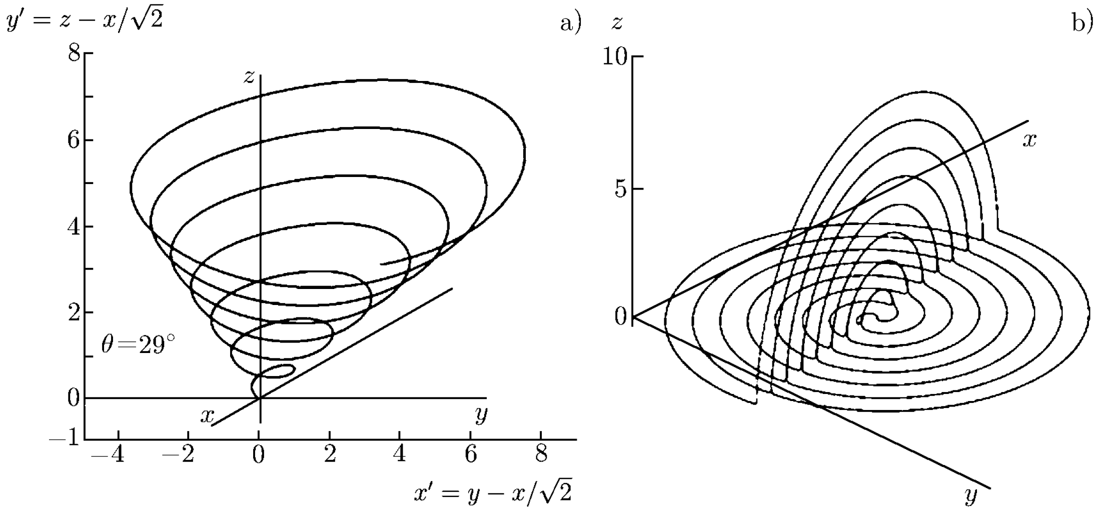

Fig. 2.

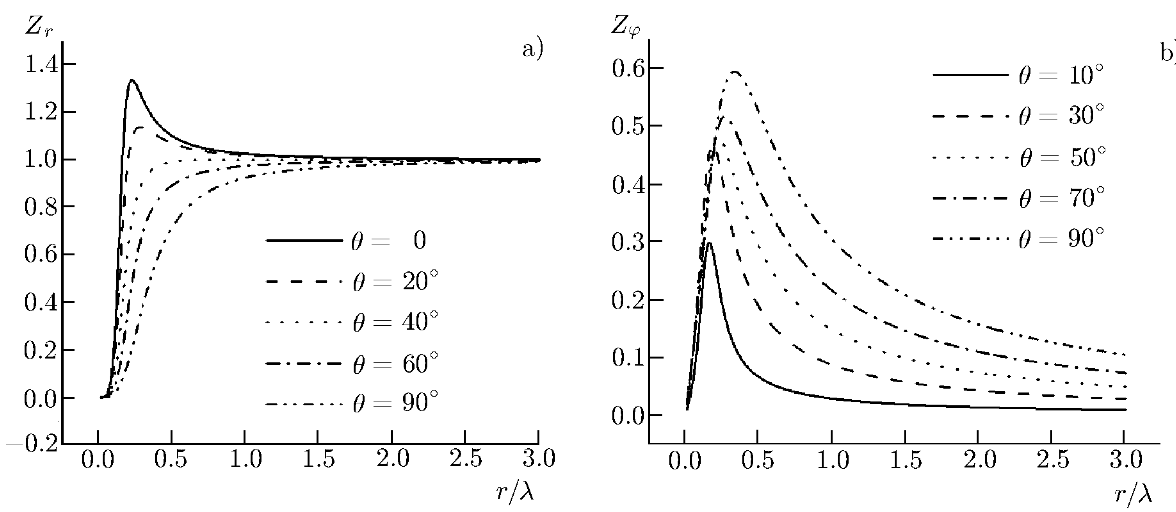

Fig. 3.

Figure 3 shows the components $\mathbf{Z}$ of the vector as functions of r for various values of the polar angle θ (reckoned from the rotation axis).

Figure 4 shows the value (averaged over the dipole-rotation period) of the cosine of the angle between the electric and magnetic fields as a function of r. It is obvious from Fig. 4 that the charged particles undergo maximum acceleration at small distances from the center of the dipole ($r < 0.2$) in the vicinity of its rotation axis ($\theta \sim 0$).

3. MOTION OF ELECTRICALLY CHARGED PARTICLES IN AN ELECTROMAGNETIC FIELD

In the nonrelativistic case, motion of charged particles in the mutually orthogonal electric and magnetic fields is represented as the circular and drift motions of the circle center [4]. However, in the relativistic case, the situation is drastically changed. First of all, the circular motion becomes elliptic for $|\mathbf{E}| < |\mathbf{B}|$ and hyperbolic for $|\mathbf{E}| > |\mathbf{B}|$. Some examples of relativistic motions can be found in [5].

The law of conservation of momentum $\mathbf{p}$ is the same for both relativistic and nonrelativistic cases:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\,(\mathbf{E} + [\mathbf{v}\mathbf{B}]/c),$$

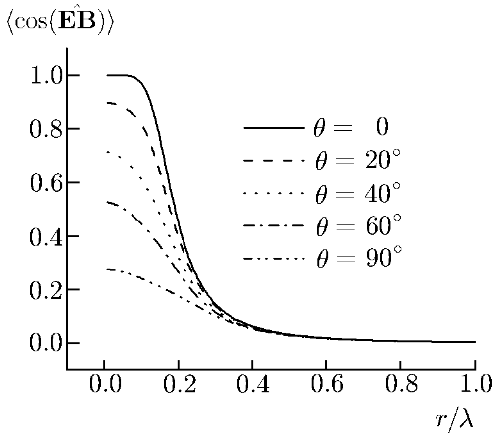

Fig. 4.

where e is the particle charge. In the dimensionless form, this expression is written as

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau'} = \mathbf{e} + [\mathbf{Ub}]/U_0. \tag{6}$$

Here, $(U_0, \mathbf{U})$ is the dimensionless 4-velocity. Its components are determined by the speed of light c and the dimensional velocity $\mathbf{v}$ of a particle with the help of the expressions

$$U_0 = 1/\sqrt{1-\beta^2}, \qquad \mathbf{U} = \mathbf{v}U_0/c, \qquad \beta = |\mathbf{v}|/c.$$

The dimensionless time is introduced as $\tau' = \Omega t$. Here $\Omega = eB_0/(mc)$ is the cyclotron frequency, where B_0 is the magnetic-field value typical of this problem, and m is the particle mass. The dimensionless fields $\mathbf{e}$ and $\mathbf{b}$ in Eq. (7) are determined by the relations $\mathbf{e} = \mathbf{E}/B_0$ and $\mathbf{b} = \mathbf{B}/B_0$, respectively.

Generally, the particle-motion trajectory is found by numerical solution (using the Runge–Kutta method) of a system of six ordinary differential equations of the first order, three of which represent the law of conservation of momentum (see Eq. (6)), while other three equations are determined by the derivatives of the coordinates

$$\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}\tau'} = \frac{\mathbf{U}}{U_0}. \tag{7}$$

Here, $\mathbf{r}' = \Omega\mathbf{r}/c = \mathbf{r}/\rho_L$ is the dimensionless radius vector, $\mathbf{r}$ is the dimensional radius vector, and $\rho_L = c/\Omega$ is the length unit which is equal to the Larmor radius if the particle would gyrate with light velocity.

If the electromagnetic field is constant and uniform, then the system of equations (6) and (7) can be solved analytically. To this end, we should pass to the coordinate system in which the vector product $[\tilde{\mathbf{e}}\tilde{\mathbf{b}}]$ of the transformed fields is zero. In [6], an algorithm allowing one to realize the Lorentz transformations without coordinate-system rotation is described. According to that algorithm, the components of a 4-vector (X_0 and $\mathbf{X}$) are transformed as follows:

$$\tilde{X}_0 = U_0X_0 - \mathbf{UX}, \qquad \tilde{\mathbf{X}} = \mathbf{X} + \mathbf{U}\frac{\mathbf{UX}}{U_0+1} - \mathbf{U}X_0.$$

Applying this expression to the vector potential (φ and $\mathbf{A}$) and taking into account that

$$\mathbf{e}(\mathbf{r}) = -\nabla\varphi, \qquad \mathbf{b}(\mathbf{r}) = [\nabla\mathbf{A}], \qquad \mathbf{r} = \tilde{\mathbf{r}} + \mathbf{U}\left(\frac{\mathbf{Ur}}{U_0+1} + \tilde{\tau}\right),$$

we write the result of the Lorentz transformations of the fields $\mathbf{e}$ and $\mathbf{b}$ in the form

$$\tilde{\mathbf{e}} = U_0\mathbf{e} - \mathbf{U}\frac{\mathbf{eU}}{U_0+1} + [\mathbf{Ub}], \qquad \tilde{\mathbf{b}} = U_0\mathbf{b} - \mathbf{U}\frac{\mathbf{bU}}{U_0+1} - [\mathbf{Ue}]. \tag{8}$$

If $[\mathbf{eb}] \neq 0$, then a particle drifts normally to the plane of the vectors $\mathbf{e}$ and $\mathbf{b}$. Hence, the drift velocity $\mathbf{U}_\mathrm{d}$ can be found from the condition $[\tilde{\mathbf{e}}\tilde{\mathbf{b}}] = 0$. This condition holds if the drift velocity is determined by the expressions

$$\mathbf{U}_\mathrm{d} = x_\mathrm{d}\,[\mathbf{eb}], \qquad x_\mathrm{d}^2 = \frac{1}{2\,[\mathbf{eb}]^2}\left(\frac{\mathbf{e}^2+\mathbf{b}^2}{\sqrt{(\mathbf{e}^2+\mathbf{b}^2)^2 - 4\,[\mathbf{eb}]^2}} - 1\right). \tag{9}$$

Figure 5 shows the drift velocity as a function of the ratio of the electric field to the magnetic field and the angle ϑ between the directions of the vectors $\mathbf{e}$ and $\mathbf{b}$. The solid curves correspond to Eq. (9), while the dashed curves correspond to the classical definition of the drift velocity, for which $x_{\mathrm{d}} = U_0$.

Upon substitution of Eq. (9) for the drift velocity into Eq. (8), the latter is transformed as

$$\tilde{\mathbf{e}} = \mathbf{e}\,(U_0 - x_{\mathrm{d}} b^2) + x_{\mathrm{d}} \mathbf{b}\,(\mathbf{eb}),$$
$$\tilde{\mathbf{b}} = \mathbf{b}\,(U_0 - x_{\mathrm{d}} e^2) + x_{\mathrm{d}} \mathbf{e}\,(\mathbf{eb}). \qquad (10)$$

Here, the fields $\mathbf{e}$ and $\mathbf{b}$ are dimensionless.

The above transformations make it possible to reduce the solution of the problem of the charge motion in a uniform field to one of the solutions presented in [5]. Indeed, if $(\mathbf{eb}) = 0$ and $|\mathbf{e}| < |\mathbf{b}|$, then $\tilde{\mathbf{e}} = 0$ and $\tilde{\mathbf{b}} = \mathbf{b}\sqrt{b^2 - e^2}/b$. In the drift frame, a particle moves with constant velocity along the vector $\tilde{\mathbf{b}}$ and gyrates. When passing to the laboratory system, the particle acquires the drift velocity, while gyration becomes elliptic due to the Lorentz contraction. The minor semiaxis is aligned with the drift velocity and is smaller than the major semiaxis by a factor of U_0.

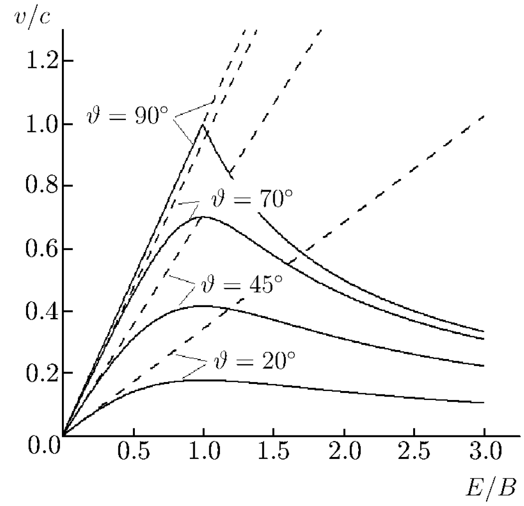

Fig. 5.

For $(\mathbf{eb}) = 0$ and $|\mathbf{e}| > |\mathbf{b}|$, in the drift system we have $\tilde{\mathbf{b}} = 0$ and $\tilde{\mathbf{e}} = \mathbf{e}\sqrt{e^2 - b^2}/e$. The problem is reduced to motion in a uniform electric field. If $(\mathbf{eb}) \neq 0$, then the charge in a system drifting with velocity $\mathbf{U}_{\mathrm{d}}$ moves in the parallel electric and magnetic fields.

4. MOTION OF ELECTRICALLY CHARGED PARTICLES IN THE ELECTROMAGNETIC FIELD OF A ROTATING DIPOLE

The study of charged-particle motions in the electromagnetic field of a rotating dipole is based on the Runge–Kutta numerical solution of the system of differential equations (6) and (7) whose right-hand sides are determined by the components of an electromagnetic field varying in space and time according to Eq. (3). The ratios of the spatiotemporal scales of the motion and field equations are determined by the quantity Ω/ω, where $\Omega = eB_0/(mc)$ is the cyclotron frequency and ω is the dipole rotation frequency. Calculations were performed for two values of the above ratio, namely, $\Omega/\omega = 9.2\cdot 10^{16}$ and $\Omega/\omega = 5\cdot 10^{13}$. The first value was used for calculating the electron motion, while the second value was used for calculating the proton motion. The corresponding value of the magnetic field was $B_0 = 10^{12}$ G and the dipole rotation frequency was assumed equal to 30 Hz. Such a high ratio of the frequencies makes us perform calculations using double accuracy. In addition, during calculations of the particle trajectories, the time step was determined by the value of the cyclotron frequency Ω. Since the electromagnetic field in the vicinity of the particle varies only slightly during one period of Larmor gyration, the character of the particle motion described in the previous section mainly remains intact.

We calculated the trajectories of the electrons and protons which were injected with nonrelativistic velocities in the vicinity of the sphere of a small radius $r_0 = 0.00101$. Initial values of the azimuthal angles were $\varphi_1 = -\pi r_0$ and $\varphi_2 = \pi/2 - \pi r_0$. For each value, the polar angles θ varied in the intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ for electrons and protons, respectively. The tracking of the particle motion was stopped once one of the two following conditions was satisfied: $r < r_{\mathrm{p}}$ (the dimensional value $r'_{\mathrm{p}} = 10$ km corresponds to the value $r_{\mathrm{p}} = 0.001$ for the chosen system of units) or $r > 1$. The initial value r_0 was taken to be greater than the value of r_{p} by 1% to rule out the possibility of the particle penetration into the sphere with radius r_{p} in the first Larmor rotation. The values of φ_1 and φ_2 corresponded to two mutually perpendicular meridional planes. The features of the particle motion were different in these planes.

Figure 6*a* shows the Lorentz factor, which determines particle acceleration, as a function of the polar angle θ of injection and the azimuthal angle φ (for protons, the polar angle equals $180° - \theta$). The numbers of the curves correspond to the subscript of the azimuthal angle φ. As is obvious from Fig. 6, the particles

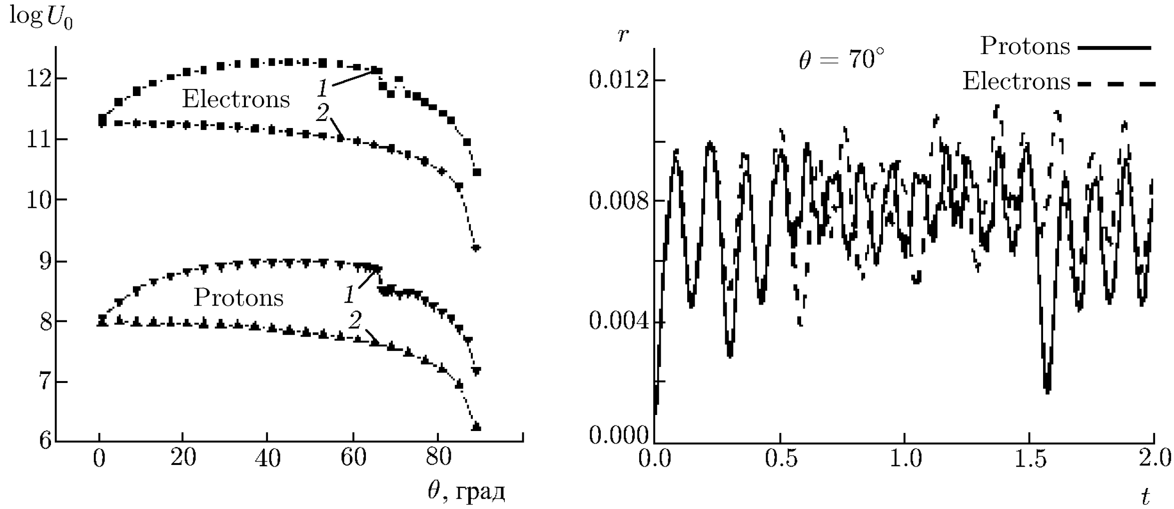

Fig. 6.

a) b)

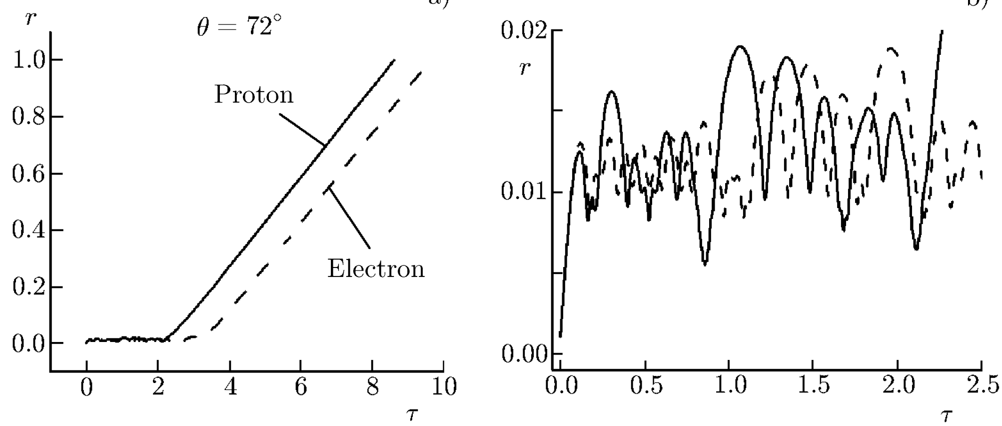

Fig. 7.

starting from the meridian φ_2 are accelerated to lower values of U_0 than the particles starting from the meridian φ_1. However, this is not the only difference. The particles with the initial azimuthal angle φ_1 and the initial polar angle $\theta < 68°$ first move increasing the distance r from the dipole to a certain value $r_{\max}$ and then move back until they reach the surface $r = r_{\text{p}}$.

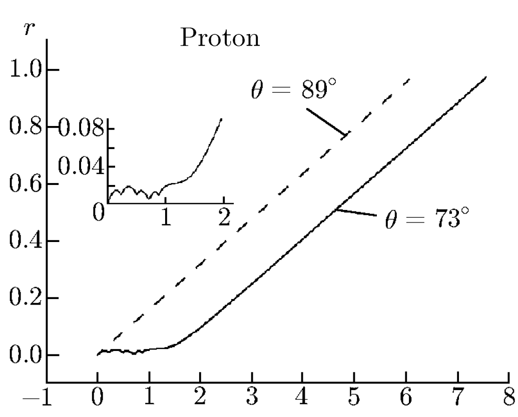

Fig. 8.

The particles with the initial polar angle close to $70°$ (with an accuracy of $1°$–$2°$) form "radiational belts" in the vicinity of the sphere of radius $r \approx 0.008$. Here, they participate in three types of motion, i.e., cyclotron rotation, latitudinal drift, and oscillations between the mirror points. For such trajectories, Fig. 6b shows the radial coordinate as a function of the dimensionless time $t = \tau\omega/\Omega$ (dipole rotates by one radian for $t = 1$).

The particles injected for $\theta > 72°$ are delayed in the radiation belts during the time that is shorter than the dipole rotation period and then move increasing monotonically the distance from the dipole. An example of such motion is shown in Figs. 7 and 8.

5. CONCLUSIONS

In the classical work [3], the nature of the electric field of neutron stars is explained by the unipolar-inductor mechanism. To ensure effective operation of the unipolar machine, the directions of the rotation and magnetic-moment axes should coincide. If these directions do not coincide, there appears a rotational electric field which cannot be compensated by polarization. Moreover, in the near region of the rotating dipole, there exist regions in which the strength of the rotational electric field can exceed the magnetic-field strength, and these fields are not orthogonal. Under such conditions, the electrically charged particles can acquire relativistic energies.

The calculated values of the Lorentz factor $U_0 = 10^{12}$ for electrons and $U_0 = 10^9$ for protons correspond to the energies of these particles of about 10^{18} eV. Allowing for the radiative loss, we can reduce the above value by several orders of magnitude.

This work was performed within the framework of the Analytical Program "Development of Research Potential of Higher School (2009–2010) of the Ministry of Education of the Russian Federation" and the Russian Federal Programs "Studies and Development in High-Priority Directions of Progress of the Research-and-Technology Complex of Russia for 2007–2012" and "Research and Pedagogical Workers for Innovative Russia (2009–2013)."

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