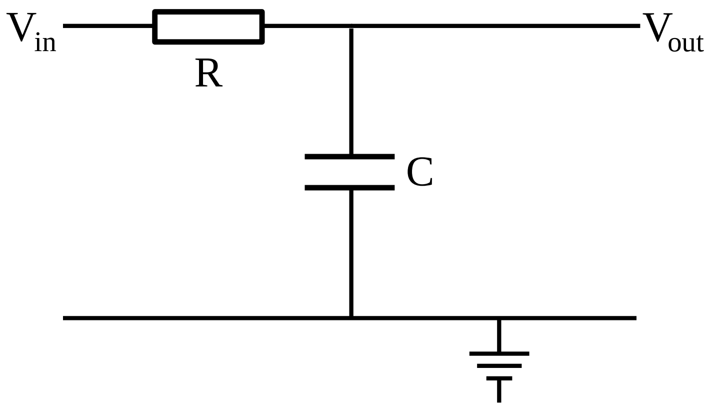
Convolution and convolution theorem

- In this lecture we will:
  - Motivate introduction of convolution by looking at the effect of an RC circuit on a signal.
  - Look at another example of convolution.
  - Introduce the (Fourier) convolution theorem.

- A comprehension question for this lecture:
  - Calculate the convolution of the functions:
    - $f(t) = \cos \omega t.$
    - $g(t) = \exp[-t].$

Effect of RC circuit on signal

- Consider low pass filter consisting of resistance R and capacitance C.



- Current through resistor $I_R = \dfrac{V_{in} - V_{out}}{R}$.
- Current through capacitor $I_C = C\dfrac{dV_{out}}{dt}$.
- These must be same, so:

$$\frac{V_{in} - V_{out}}{R} = C\frac{dV_{out}}{dt}.$$

- Rewriting: $\dfrac{dV_{out}}{dt} + \dfrac{1}{RC}V_{out} = \dfrac{V_{in}}{RC}$.
- Solve using integrating factor:

$$IF = \exp\left[\int \frac{1}{RC}dt\right] = \exp\left[\frac{t}{RC}\right].$$

- Multiplying through by the IF:

$$\frac{dV_{out}}{dt}e^{t/RC} + \frac{1}{RC}V_{out}e^{t/RC} = \frac{V_{in}}{RC}e^{t/RC},$$

$$\text{so } \frac{d}{dt}\left(V_{out}e^{t/RC}\right) = \frac{V_{in}}{RC}e^{t/RC}$$

$$\text{and } V_{out}(t)e^{t/RC} = \frac{1}{RC}\int V_{in}(t)e^{t/RC}\,dt$$

$$\text{or } V_{out}(t) = \frac{e^{-t/RC}}{RC}\int V_{in}(t)e^{t/RC}\,dt.$$

Effect of RC circuit on signal – convolution

- Introducing a dummy variable and limits for the integration:

$$V_{out}(t) = \frac{e^{-t/RC}}{RC}\int_{-\infty}^{t} V_{in}(\tau)\, e^{\tau/RC}\, d\tau$$

$$= \frac{1}{RC}\int_{-\infty}^{t} V_{in}(\tau)\, e^{-(t-\tau)/RC}\, d\tau.$$

- The result of sending a signal $V_{in}(t)$ through the filter with response function $r(t) = e^{-t/RC}/RC$ is given by the *convolution* of V_{in} and r:

$$V_{out}(t) = \left(V_{in} * r\right)(t)$$

$$= \int_{-\infty}^{t} V_{in}(\tau)\, r(t-\tau)\, d\tau$$

$$= \frac{1}{RC}\int_{-\infty}^{t} V_{in}(\tau)\, e^{-(t-\tau)/RC}\, d\tau.$$

- You will also see this written:

$$V_{out}(t) = V_{in}(t) * r(t).$$

- Determine output if $V_{in}(t) = V_0 \sin(\omega t)$.
- We will set $V_0 = R = C = 1$ to simplify things!

$$V_{out}(t) = \int_{-\infty}^{t} \sin\omega\tau\, e^{-(t-\tau)}\, d\tau$$

- Integrate by parts once…

$$V_{out}(t) = -\int_{-\infty}^{t} e^{-(t-\tau)}\, d\left(\frac{\cos\omega\tau}{\omega}\right)$$

$$= -\frac{\cos\omega\tau}{\omega} e^{-(t-\tau)}\Big|_{-\infty}^{t}$$

$$+\int_{-\infty}^{t} \frac{\cos\tau}{\omega} e^{-(t-\tau)}\, d\tau$$

Effect of RC circuit on signal – convolution

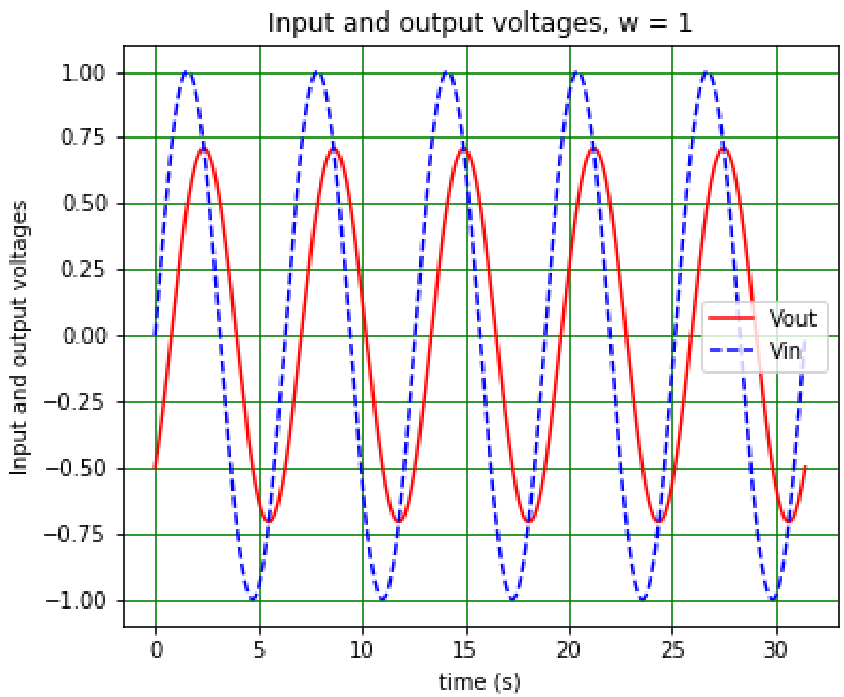
- $$V_{out} = -\frac{\cos\omega\tau}{\omega} e^{-(t-\tau)}\Big|_{-\infty}^{t} + \int_{-\infty}^{t} \frac{\cos\omega\tau}{\omega} e^{-(t-\tau)}\, d\tau = -\frac{\cos\omega\tau}{\omega} + \int_{-\infty}^{t} \frac{\cos\omega\tau}{\omega} e^{-(t-\tau)}\, d\tau$$

- Integrate by parts again:

$$V_{out} = -\frac{\cos\omega t}{\omega} + \int_{-\infty}^{t} e^{-(t-\tau)}\, d\left(\frac{\sin\omega\tau}{\omega^2}\right) = -\frac{\cos\omega t}{\omega} + \frac{\sin\omega\tau}{\omega^2} e^{-(t-\tau)}\Big|_{-\infty}^{t} - \int_{-\infty}^{t} \frac{\sin\omega\tau}{\omega^2} e^{-(t-\tau)}\, d\tau$$

$$= -\frac{\cos\omega t}{\omega} + \frac{\sin\omega t}{\omega^2} - \frac{1}{\omega^2}\int_{-\infty}^{t} \sin\omega t\, e^{-(t-\tau)}\, d\tau$$

$$= -\frac{\cos\omega t}{\omega} + \frac{\sin\omega t}{\omega^2} - \frac{V_{out}}{\omega^2}$$



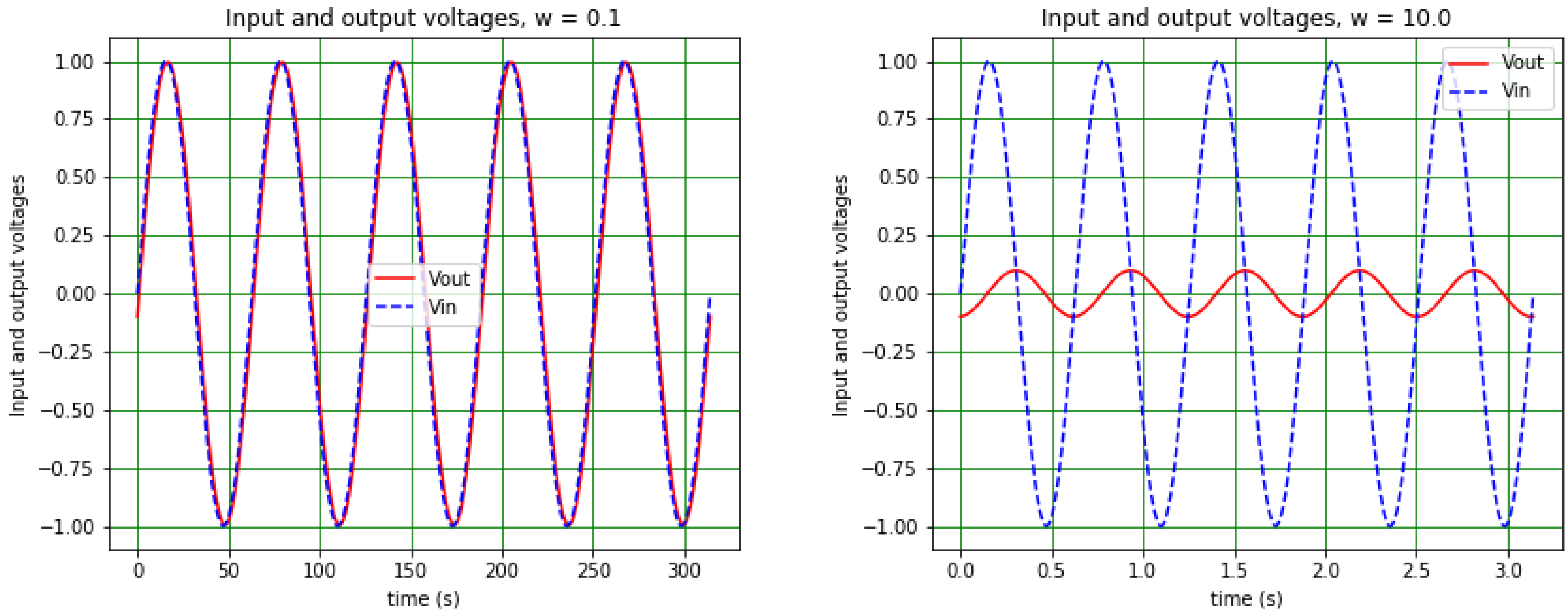
- Hence:

$$\left(1 + \frac{1}{\omega^2}\right) V_{out} = \frac{\sin\omega t}{\omega^2} - \frac{\cos\omega t}{\omega}$$

$$V_{out} = \frac{\sin\omega t - \omega\cos\omega t}{1+\omega^2}$$

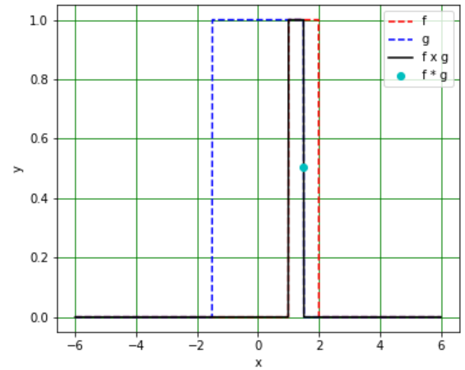
Effect of RC circuit on signal – convolution

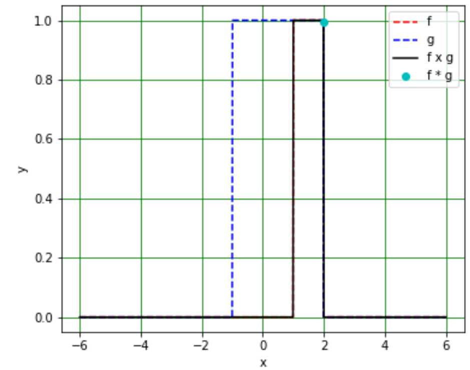
- Look at response of circuit at low and high frequencies:

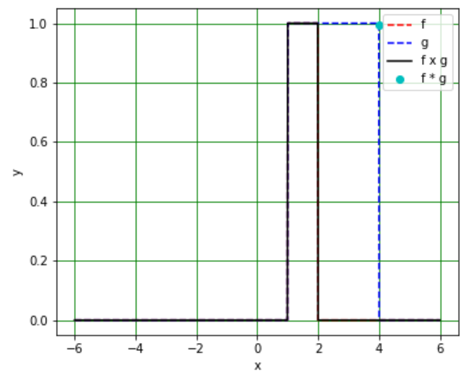


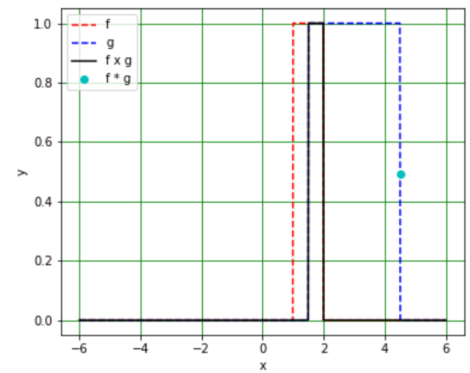
- See amplitude change, but also in V_{in} and V_{out} in phase for $\omega << 1$, V_{out} lags behind V_{in} by $\pi/4$ for $\omega = 1$ and by $\pi/2$ for $\omega >> 1$.

Convolution example









- Look at functions:

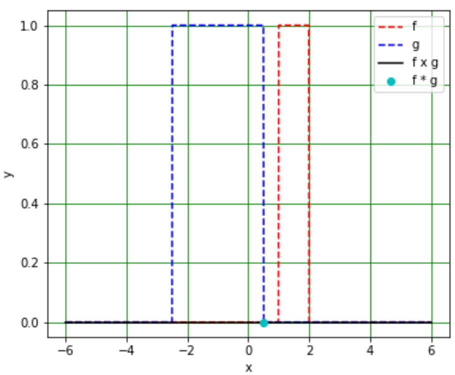
$$f(x) = \begin{cases} 1 \text{ if } 1 < x < 2 \\ 0 \text{ otherwise.} \end{cases}$$

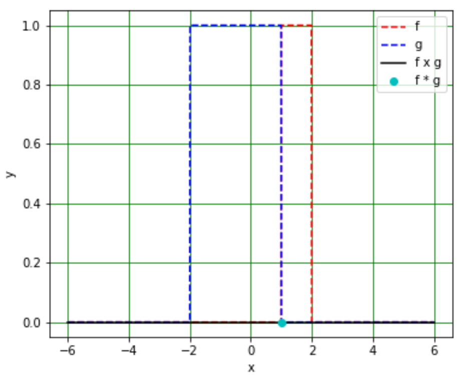
$$g(x) = \begin{cases} 1 \text{ if } 0 < x < 3 \\ 0 \text{ otherwise.} \end{cases}$$

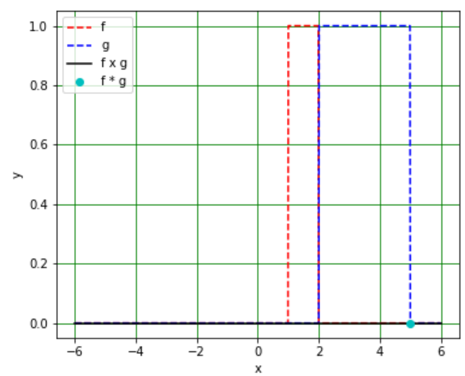
- And their convolution:

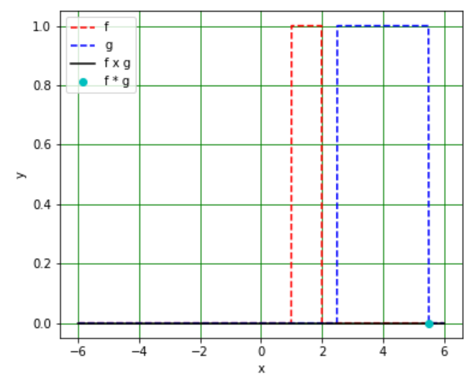
$$(f * g)(x) = \int_{-\infty}^{\infty} f(\xi) g(x - \xi) d\xi.$$

- For x = 0.5, 1, 1.5, 2, 4, 4.5, 5, 5.5, we have (L to R, top to bottom):



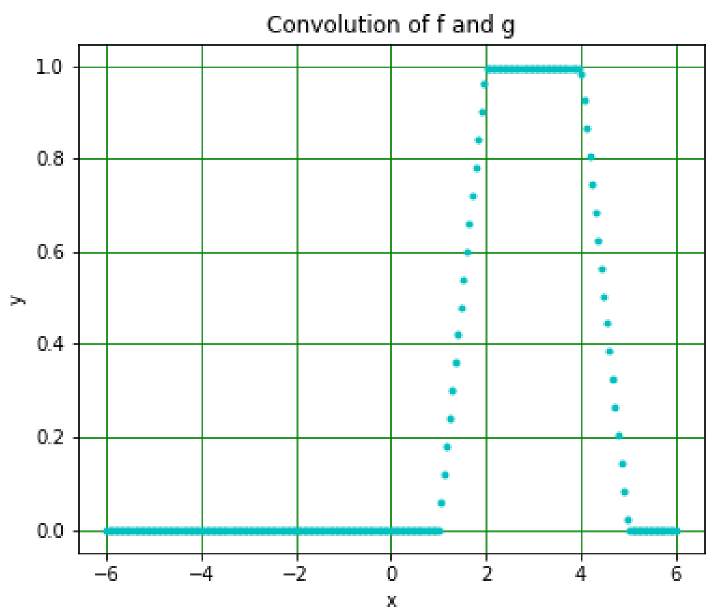






Convolution example

- The value of $(f*g)(x)$ at a given x is the overlapping area of f and g with $g(\xi)\to g(-\xi)$.
- Putting the graphs on the previous slide together, $(f*g)(x)$ is:



Convolution theorem

- If $\mathcal{F}(f)$ is the Fourier Transform of f and $\mathcal{F}(g)$ that of g, then: $\mathcal{F}(f*g)=\mathcal{F}(f)\mathcal{F}(g)$.
- Using the inverse Fourier Transform, we can write: $f*g=\mathcal{F}^{-1}\left(\mathcal{F}(f)\mathcal{F}(g)\right)$.