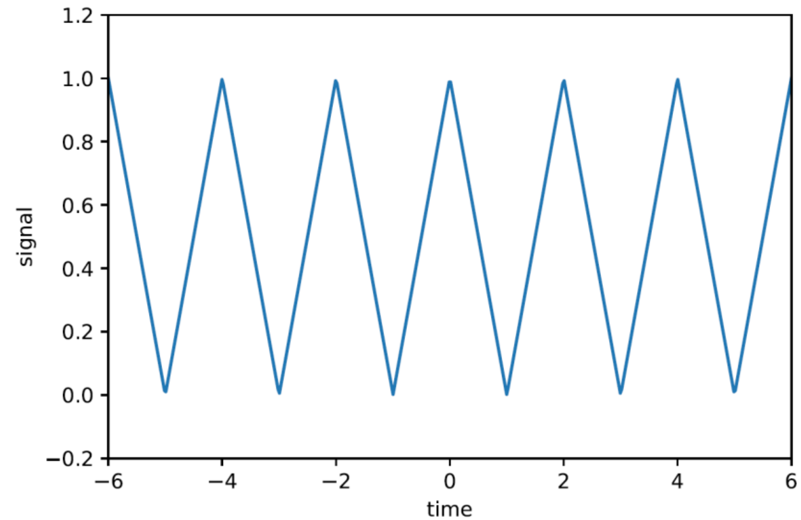
Filters and forced oscillations – Fourier series in physics

- In this lecture we will:
  - See a practical use of Fourier series in analysing electronic circuits.
  - See how 2nd order differential equations can arise in physical situations such as the motion of masses on springs.
  - Examine the case of periodic "forcing terms" and see how to deal with them using Fourier series.
  - Do an example.
- A comprehension question for this lecture:
  - Deduce as much as you can about the coefficients in the Fourier series for the following function:



  - Compare your guesses with the true values.

Fourier series in practical physics

- Suppose we have two electronic circuits that only let through signals in certain frequency ranges:
  - $f < f_{top}$ (a "low-pass" filter).
  - $f > f_{bot}$ (a "high-pass" filter).
- What will we see if we send a square wave signal through these circuits?
- Input:

$$f(t) = \begin{cases} -1 \text{ if } -2 \le t < -1 \\ 1 \text{ if } -1 \le t < 1 \\ -1 \text{ if } 1 \le t < 2 \end{cases}.$$

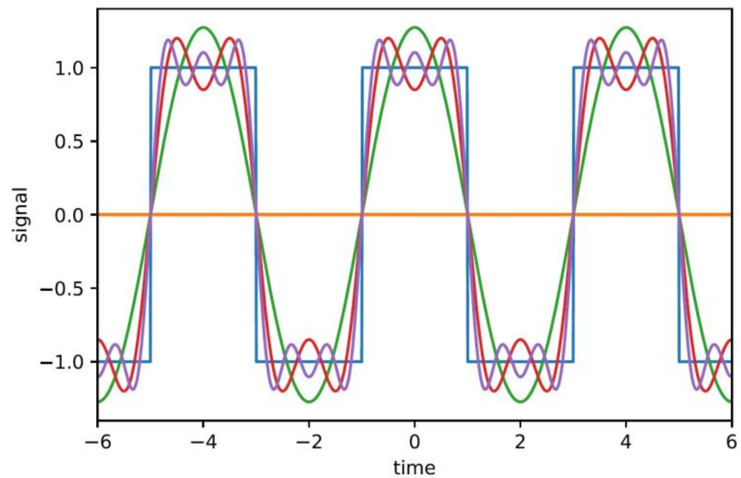
- Represent as a Fourier series.
- Show that:

$$a_0 = 0$$

$$b_n = 0$$

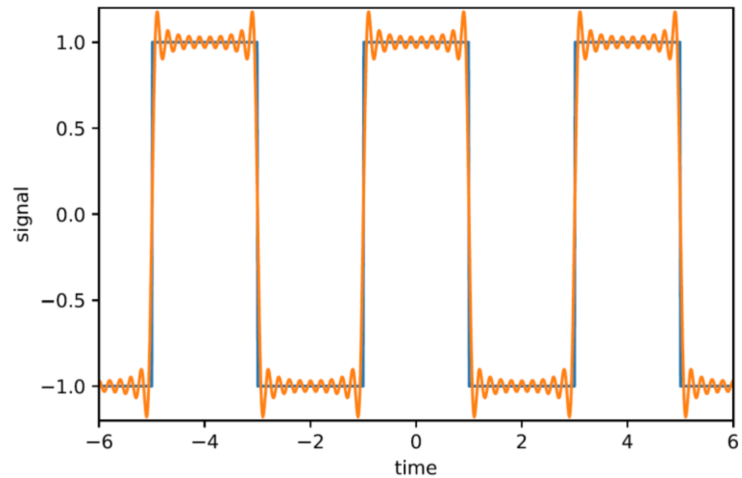
$$a_n = \frac{2}{n\pi} \sin \frac{n\pi}{2}.$$

- First few terms:



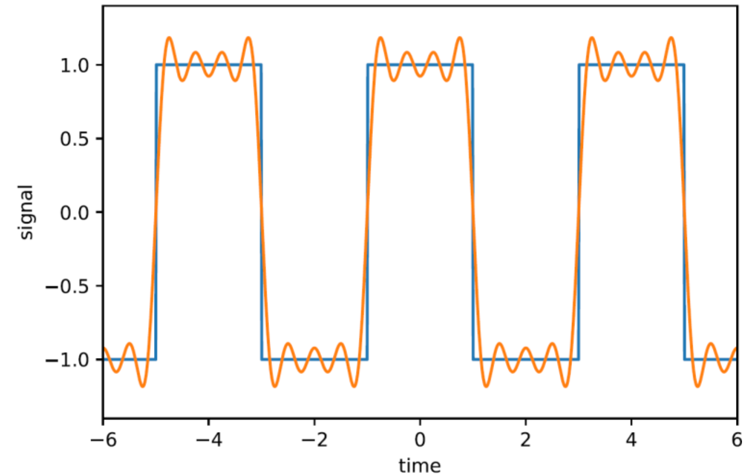
Why use Fourier Series?

- If add up first 20 terms get reasonable representation of input:

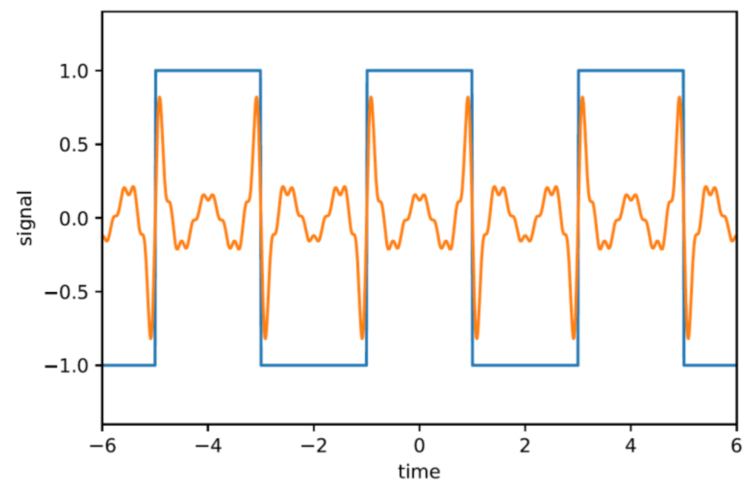


- What do we get if we pass this signal through our low- and high-pass filters?
- Find out by applying effect of circuit to sine and cosine terms that make up input, then adding them up again.

- Low-pass (cut off terms above tenth):



- High-pass (remove terms below fourth):

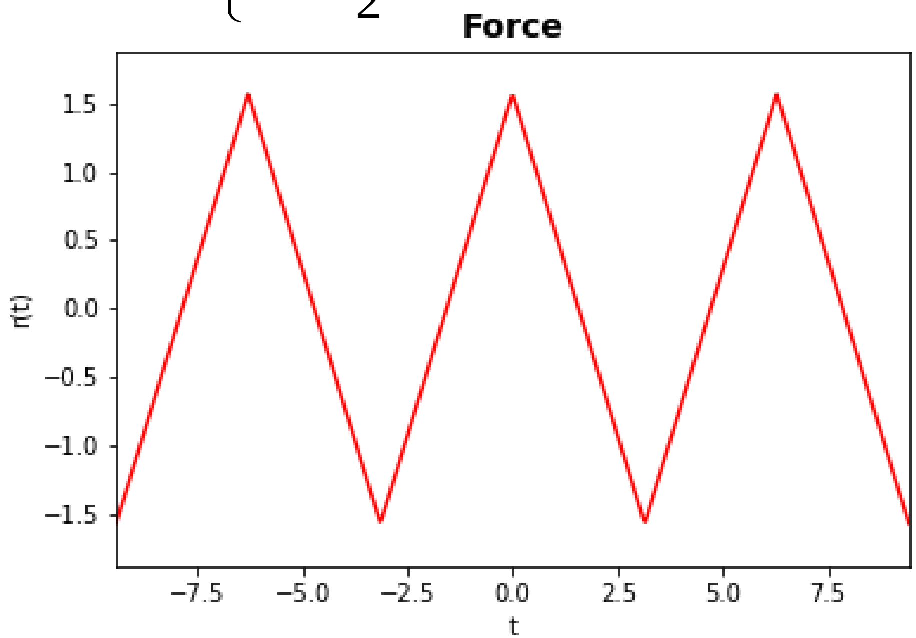


Forced oscillations

- Consider a mass m attached to a spring with spring constant k.
- The force on the mass, at distance y from equilibrium, is $F = -ky$.
- Newton's second law relates the force to the acceleration, $F = m\ddot{y}$.
- Hence $m\ddot{y} = -ky$.
- Now assume that an external force r(t) is also applied to the mass.
- Then: $m\ddot{y} = r(t) - ky$ or

  $$m\ddot{y} + ky = r(t).$$
- We have seen how to solve this if r(t) is something like $r(t) = \cos\gamma t + \sin\gamma t$.
- What if r(t) is a more complicated periodic function?
- Can solve by representing r(t) as a Fourier series.
- An example: $m = 1,\ k = 4$ and

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{for } -\pi < t \le 0 \\ -t + \dfrac{\pi}{2} & \text{for } 0 < t \le \pi \end{cases}.$$



Forced oscillations

- First compute the Fourier series for r(t).
- Only cosine terms (even function), a_0 is zero (average of r(t)) and $T = 2\pi$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} r(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\left(t+\frac{\pi}{2}\right)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}\left(-t+\frac{\pi}{2}\right)\cos nt\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left(-t+\frac{\pi}{2}\right)\cos nt\,dt$$

$$= \frac{2}{\pi}\left(-\int_{0}^{\pi} t\,d\left(\frac{\sin nt}{n}\right) + \frac{\pi}{2}\int_{0}^{\pi}\cos nt\,dt\right)$$

$$= \frac{2}{\pi}\left(\left[-t\frac{\sin nt}{n}\right]_0^{\pi} + \int_{0}^{\pi}\frac{\sin nt}{n}dt + \frac{\pi}{2}\left[\frac{\sin nt}{n}\right]_0^{\pi}\right)$$

- Hence:

$$a_n = \frac{2}{\pi}\left[-\frac{\cos nt}{n^2}\right]_0^{\pi}$$

$$= \frac{2(1-\cos n\pi)}{\pi n^2}.$$

- Now $1-\cos n\pi$ is 2 if n is odd and zero if n is even, so:

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \ldots\right).$$

- The term in $\cos nt$ in the series for r(t) is:

$$\frac{4}{n^2\pi}\cos nt.$$

Forced oscillations

- Look at the original equation for this $\cos nt$ term:

$$\ddot{y} + 4y = \frac{4}{n^2\pi}\cos nt.$$

- The particular integral is of the form:

$$y_n = A_n \cos nt + B_n \sin nt$$

$$\Rightarrow \dot{y}_n = -nA_n \sin nt + nB_n \cos nt$$

$$\text{and } \ddot{y}_n = -n^2 A_n \cos nt - n^2 B_n \sin nt.$$

- Equating coefficients gives:

$$A_n = \frac{4}{n^2\pi(4-n^2)} \text{ and } B_n = 0.$$

- Since the complete force term is the sum of $\cos nt$ terms for $n = 1, 3, 5\ldots$ the full particular integral y_p will be the sum of terms y_1, y_3, $y_5\ldots$
- That is:

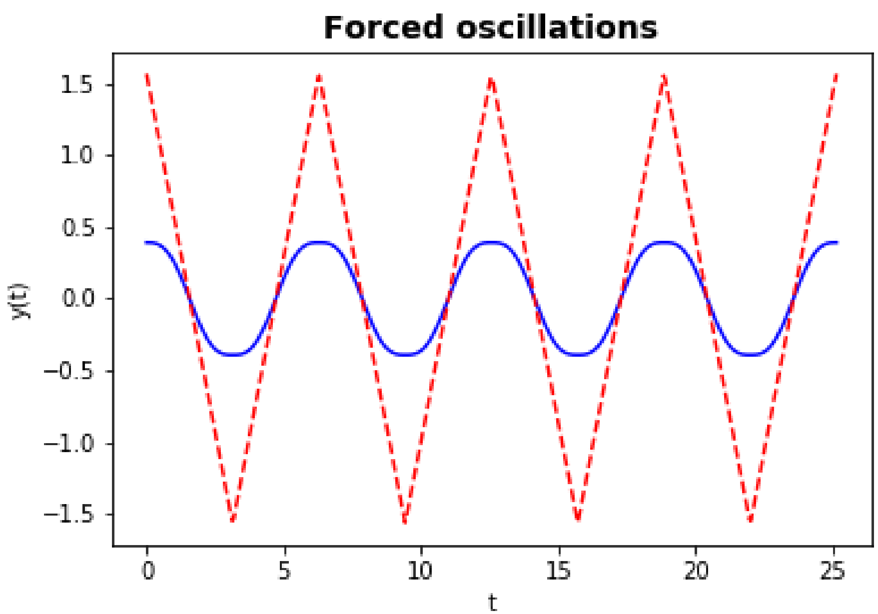
$$y_p = \frac{4}{\pi}\left(\frac{\cos t}{1^2(4-1^2)} + \frac{\cos 3t}{3^2(4-3^2)} + \frac{\cos 5t}{5^2(4-5^2)}\ldots\right)$$

- The solution of the homogeneous equation is: $y_c = A\cos 2t + B\sin 2t.$
- The full solution is $y = y_c + y_p$:

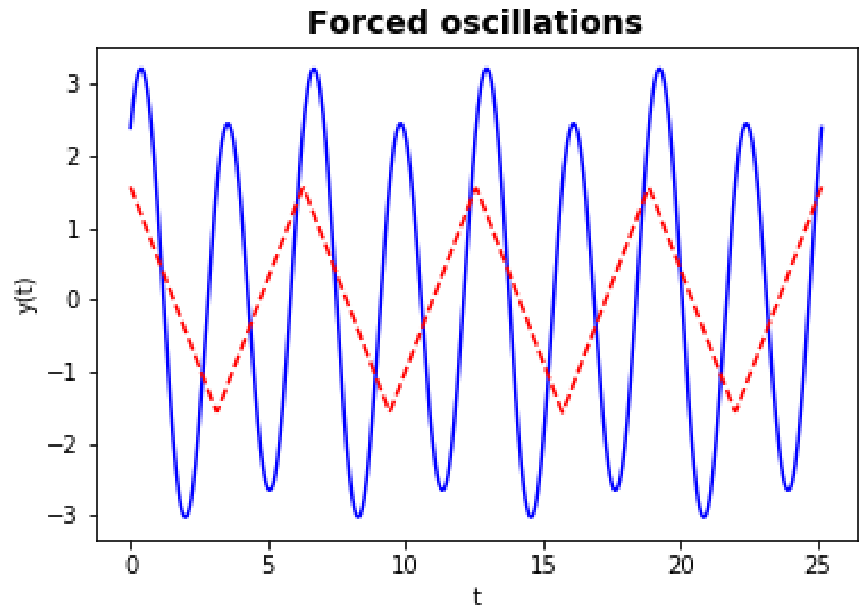
$$y = A\cos 2t + B\sin 2t + \frac{4}{\pi}\left(\frac{\cos t}{1^2(4-1^2)} + \frac{\cos 3t}{3^2(4-3^2)} + \frac{\cos 5t}{5^2(4-5^2)}\ldots\right).$$

Forced oscillations

- We see the force term excites a spectrum of oscillations with amplitudes that decrease with frequency.
- There is no friction; initial conditions influence the oscillations for all t.
- If motion due to force only:



- Motion including component due to a particular initial position and velocity.



- Friction would cause component due to initial motion to die out, leaving only that due to the force.

Resonance

- What happens if we change the spring constant?
- If the chosen value means that the natural frequency of the system is the same as one of the frequencies in the force term, resonance occurs.
- E.g. pick $k = 25$.
- Then $y_c = A\cos 5t + B\sin 5t$.
- In the particular integral, we now have to use $y_5 = A_5 t\cos 5t + B_5 t\sin 5t$, as y_c already contains $\cos 5t$ and $\sin 5t$ terms.
- We can see this frequency component ("mode") has an amplitude that grows with time, there is a "resonance".
- If there is no (or only little) friction, this mode can become large: the results can be quite interesting!
- Tacoma narrows bridge collapse.