Fourier series

- In this lecture we will:
  - See how to represent functions with period 2π using Fourier series.
  - Derive expressions for the coefficients in these series.
  - Do an example.
- A comprehension question for this lecture:
  - Show that:

$$\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = 0 \text{ if } m \neq n.$$

Fourier series

- Fourier series are a basic tool for representing periodic functions.
- A function is periodic, with period T, if $f(t+T) = f(t)$.
- For example, cosines and sines have period 2π: $\cos t = \cos(t + 2\pi)$,
  and $\sin t = \sin(t + 2\pi)$.
- If $f(t) = f(t + 2\pi)$, we can write the function as a sum of cosine and sine terms:

$$f(t) = a_0 + a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

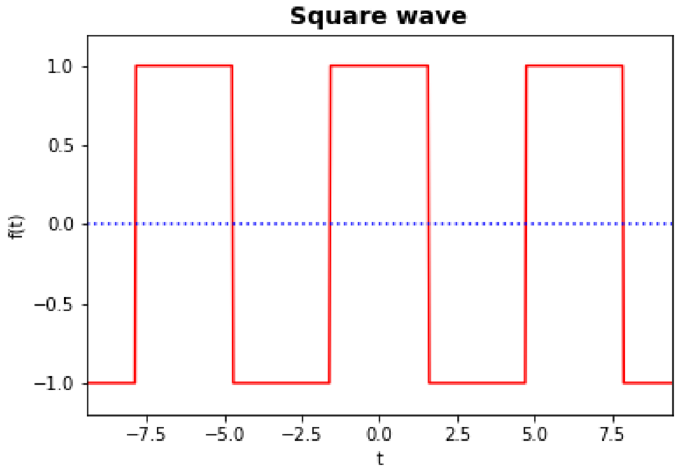
- Each term has period 2π, so this is also true for the sum.
- Look at an example, a square wave:

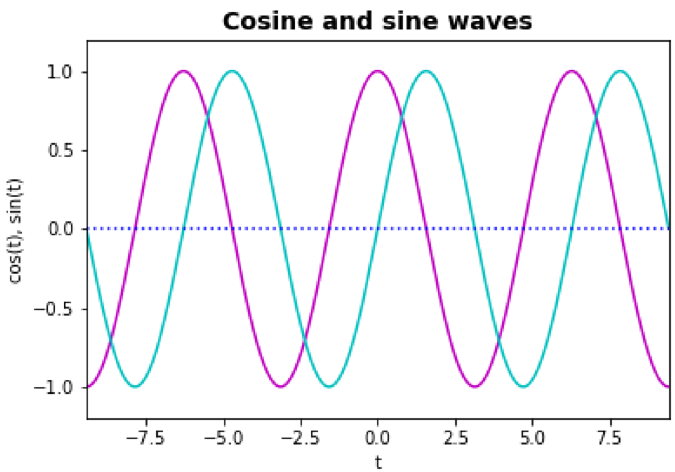
$$f(t) = \begin{cases} -1 \text{ if } -\pi \le t < -\frac{\pi}{2} \\ 1 \text{ if } -\frac{\pi}{2} \le t < \frac{\pi}{2} \\ -1 \text{ if } \frac{\pi}{2} \le t < \pi \end{cases}.$$

- First try and visualise how this can be represented as a sum of cosines and sines.

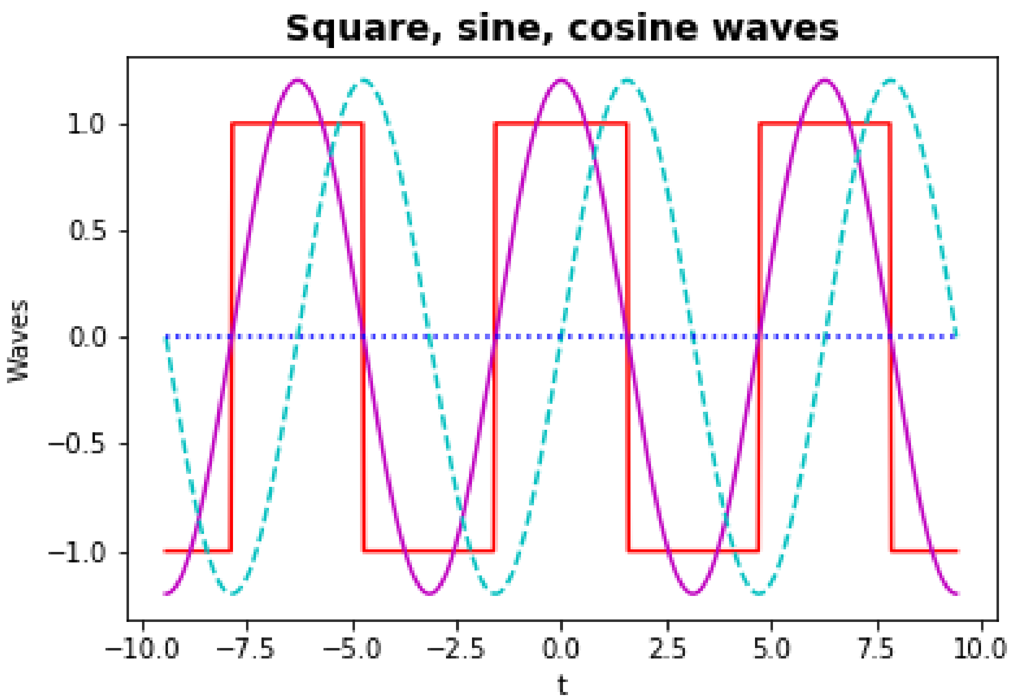
Fourier series of square wave by inspection

- Square, cosine and sine waves:





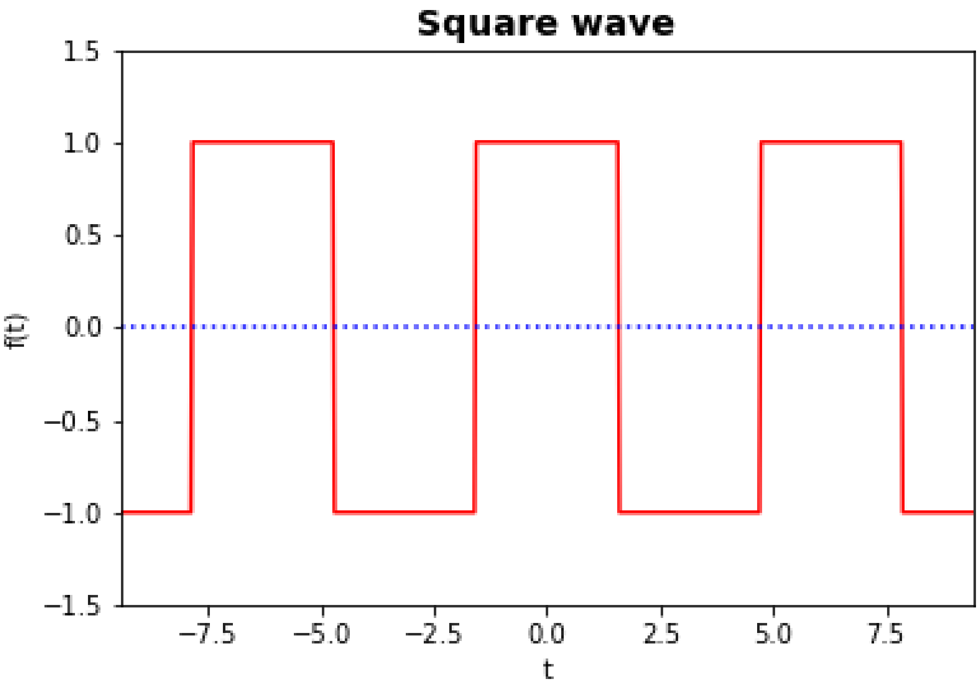
- See cosine can form approx. to this square wave, but sine doesn't work:



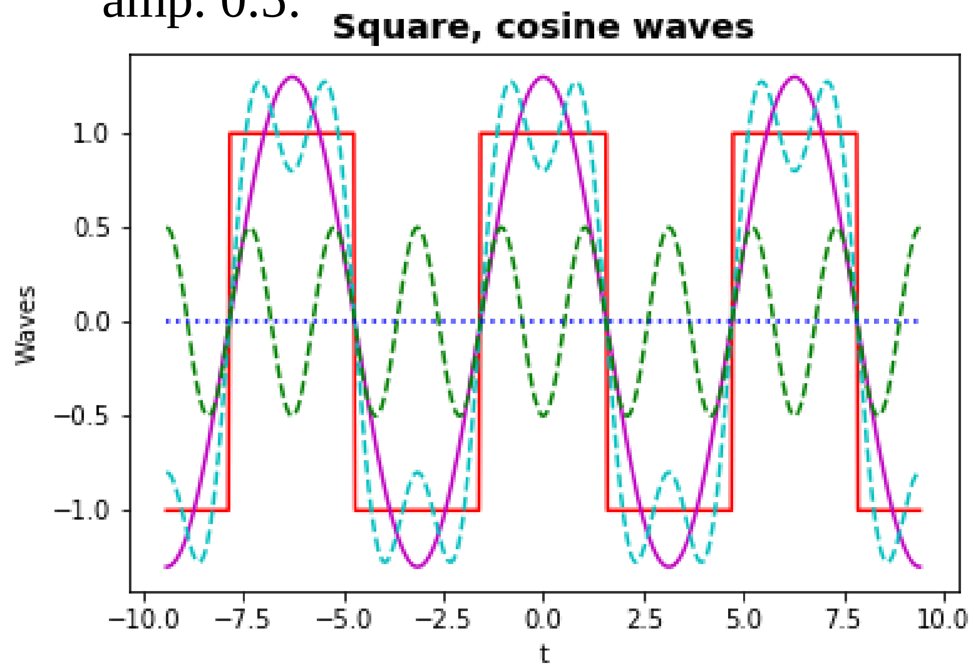
- (Amplitude of cosine above is 1.3.)
- To get better approx. need to flatten peaks and sharpen flanks of cosine.

Fourier series of square wave by inspection

- Sketch the cosine curve with freq. 1 (period 2π) on the graph, below.
- Can adding a cosine with freq. 2 (i.e. period π) and appropriate amplitude improve the approximation?



- Try adding a cosine with freq. 3, amp. 0.5:



- Looks better!
- How can we determine the freq.s and amplitudes of the cosines and sines that give the best approximation?

Determining Fourier coefficients

- Use the fact that cosines and sines form a set of orthonormal basis functions:

$$\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = 0, \text{ for } n \neq m,$$

$$\int_{-\pi}^{\pi} \sin nt \sin mt \, dt = 0, \text{ for } n \neq m,$$

$$\int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0, \text{ for } m \neq n,$$

$$\int_{-\pi}^{\pi} \cos^2 nt \, dt = \int_{-\pi}^{\pi} \sin^2 nt \, dt = \pi.$$

- Example proof one:

$$\int_{-\pi}^{\pi} \sin nt \cos mt \, dt =$$

$$\int_{-\pi}^{\pi} \tfrac{1}{2}\left(\sin(n+m)t + \sin(n-m)t\right) dt$$

$$[\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta]$$

- $$\int_{-\pi}^{\pi} \tfrac{1}{2}\left(\sin(n+m)t + \sin(n-m)t\right) dt$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)t}{n+m}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[-\frac{\cos(n-m)t}{n-m}\right]_{-\pi}^{\pi}$$

- Now cosine is even, that is, $\cos(\theta) = \cos(-\theta)$, so:

$$\cos(n+m)t\Big|_{-\pi}^{\pi}$$

$$= \cos\left((n+m)\pi\right) - \cos\left(-(n+m)\pi\right)$$

$$= \cos\left((n+m)\pi\right) - \cos\left((n+m)\pi\right)$$

$$= 0.$$

- Same is true for 2nd term above.
- Hence:

$$\int_{-\pi}^{\pi} \sin nt \cos mt \, dt = 0.$$

Determining Fourier coefficients

- Example proof two:

$$\int_{-\pi}^{\pi} \cos^2 nt\, dt = \int_{-\pi}^{\pi} \tfrac{1}{2}(1+\cos 2nt)\, dt$$

$$[\cos 2\alpha = 2\cos^2\alpha - 1]$$

- Hence:

$$\int_{-\pi}^{\pi} \cos^2 nt\, dt = \frac{1}{2}\left[t + \frac{\sin 2nt}{2n}\right]_{-\pi}^{\pi}$$

$$= \pi.$$

- We can use these orthonormality properties to determine the coefficients in our Fourier series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt.$$

- E.g. multiply $f(t)$ by $\cos mt$ and integrate.

$$\int_{-\pi}^{\pi} f(t)\cos mt\, dt =$$

$$\int_{-\pi}^{\pi}\left(a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt\right)\cos mt\, dt.$$

- Every term is zero except for when $m = n$.
- In this case the "sin × cos" term is zero (odd function integrated over symmetric range) and we are left with:

$$\int_{-\pi}^{\pi} f(t)\cos mt\, dt = \int_{-\pi}^{\pi} a_m \cos^2 mt\, dt = a_m \pi$$

$$\Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos mt\, dt.$$

Determining Fourier coefficients

- Similar proofs lead to the full set of expressions needed to determine the Fourier coefficients.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt.$$

- Go back to the square wave example and work out the coefficients using the above formulae.

- First, constant term:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(-1)\,dt + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)\,dt + \frac{1}{2\pi}\int_{\pi/2}^{\pi}(-1)\,dt$$
$$= \frac{1}{2\pi}\left[-t\right]_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\left[t\right]_{-\pi/2}^{\pi/2} + \frac{1}{2\pi}\left[-t\right]_{\pi/2}^{\pi}$$
$$= 0.$$

- Also, see that a_0 is the average of the function over the interval $[-\pi, \pi]$.
- Hence, expect $a_0 = 0$ for this function.
- Function f(t) is even, so there can be no contributions from the odd sine functions, hence $b_n = 0$.

Determining Fourier coefficients

- Cosine is even, so it will contribute to series.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} -\cos nt\,dt + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\,dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} -\cos nt\,dt$$

$$= \frac{1}{\pi}\left[-\frac{\sin nt}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[-\frac{\sin nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left(-\sin\frac{-n\pi}{2} + \sin\frac{n\pi}{2} - \sin\frac{-n\pi}{2} + \sin\frac{n\pi}{2}\right)$$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}.$$

- The first few terms are:

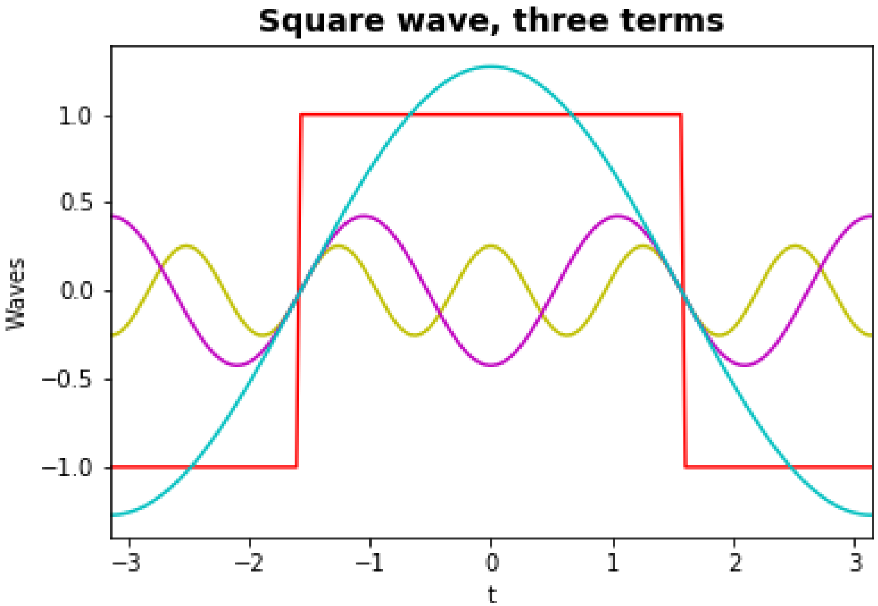
$$a_1 = \frac{4}{\pi},\ a_2 = 0,\ a_3 = -\frac{4}{3\pi},\ a_4 = 0,\ a_5 = \frac{4}{5\pi}\ldots$$

- Numerically:

n	a_n
1	1.27
2	0
3	-0.42
4	0
5	0.25

Square wave as Fourier series

- Square wave with first three non-zero Fourier terms (i.e. five terms):



- Opposite, square wave with:
  - Fourier series for square wave, first three non-zero terms.
  - Fourier series, first twenty terms.