Differential equations

- In this lecture we will:
  - Find out how to solve some more inhomogeneous second order differential equations.
  - See how some second order equations can be reduced to first order.
  - Comment on some techniques for solving general second order linear differential equations.

- Some comprehension questions for this lecture.
  - Find the general solution of the equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 7\cos 3x$$

  - Solve the equation:

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Inhomogeneous second order differential equations

- If f(x) is of the form $C\sin\gamma x$ or $D\cos\gamma x$, or a sum of these terms, the trial solution is $y_p = A\cos\gamma x + B\sin\gamma x$
- Example:
- Find the particular integral of

  $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \cos x.$$

- The auxiliary equation is

  $$m^2 - 4m + 3 = 0$$

  $$\Rightarrow (m-1)(m-3) = 0$$

  so $m_1 = 1$ and $m_2 = 3$.

- The roots are real and distinct, so the solution of the complementary equation is $y_c = C_1e^x + C_2e^{3x}$.

- $y_p = A\cos x + B\sin x,$

  $$\frac{dy_p}{dx} = -A\sin x + B\cos x,$$

  $$\frac{d^2y_p}{dx^2} = -A\cos x - B\sin x.$$

- Substituting gives...

  $$(-A\cos x - B\sin x)$$

  $$-4(-A\sin x + B\cos x)$$

  $$+3(A\cos x + B\sin x) = \cos x$$

  $$\Rightarrow -B + 4A + 3B = 0$$

  and $-A - 4B + 3A = 1.$

  Hence $B = -2A$ and $10A = 1$

  $$\Rightarrow A = 1/10 \text{ and } B = -1/5.$$

Inhomogeneous second order differential equations

- Therefore $y_p = \frac{1}{10}\cos x - \frac{1}{5}\sin x.$
- Again, if the solutions of the complementary equation are of the same form as the particular integral, the latter must be modified.
- Example:

$$\frac{d^2y}{dx^2} + 4y = \cos 2x.$$

- The auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow m_1 = 2i,\ m_2 = -2i$$

- The solution of the complementary equation is $y_c = C_1\cos 2x + C_2\sin 2x.$
- We therefore try a particular integral of the form $y_p = x(A\cos 2x + B\sin 2x).$
- Differentiating:

$$\frac{dy_p}{dx} = A\cos 2x - 2Ax\sin 2x$$
$$+ B\sin 2x + 2Bx\cos 2x$$

$$\frac{d^2y_p}{dx^2} = -2A\sin 2x - 2A\sin 2x - 4Ax\cos 2x$$
$$+ 2B\cos 2x + 2B\cos 2x - 4Bx\sin 2x$$
$$= -4A\sin 2x - 4Ax\cos 2x$$
$$+ 4B\cos 2x - 4Bx\sin 2x.$$

Inhomogeneous second order DEs

- Substituting:

$$\begin{pmatrix} -4A\sin 2x - 4Ax\cos 2x \\ \qquad + 4B\cos 2x - 4Bx\sin 2x \end{pmatrix}$$
$$+4\left(Ax\cos 2x + Bx\sin 2x\right) = \cos 2x$$
$$\Rightarrow -4A = 0 \ [\sin 2x \ \text{term}]$$
$$-4A + 4A = 0 \ [x\cos 2x \ \text{term}]$$
$$4B = 1 \ [\cos 2x \ \text{term}]$$
$$-4B + 4B = 0 \ [x\sin 2x \ \text{term}].$$

- Hence $B = \frac{1}{4}$ and $y_p = \frac{1}{4}x\sin 2x.$

- The general solution is therefore:

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{4}x\sin 2x.$$

Equation reducible to first order – type 1

- A second order equation with no explicit y dependence, i.e. of the form:

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0$$

can be reduced to a first order equation by changing the dependent variable.

- Putting $v = \frac{dy}{dx}$ gives $f\left(\frac{dv}{dx}, v, x\right) = 0.$

- This may be soluble using the methods for first order equations we have discussed previously.

Equation reducible to first order – type 1

- Example:
- Solve the initial value problem:

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2, \; y(0)=1, \; y'(0)=-2.$$

- No explicit y dependence, put $v = \dfrac{dy}{dx}$.
- Then have:

$$\frac{dv}{dx} = xv^2$$

$$\Rightarrow \frac{dv}{v^2} = x\,dx \;\text{ and }\; -\frac{1}{v} = \frac{x^2}{2} + \frac{A}{2}$$

$$\Rightarrow v = -\frac{2}{x^2 + A} \;\text{ or }\; \frac{dy}{dx} = -\frac{2}{x^2 + A}.$$

- We have y'(0) = – 2, so A = 1.
- Using this we can perform a further integration:

$$y = -2\int \frac{dx}{x^2+1}$$

$$= -2\tan^{-1} x + B.$$

- The condition $y(0) = 1$ allows the determination of B:

$$-2\tan^{-1}(0) + B = 1$$

$$\Rightarrow B = 1.$$

- Hence:

$$y = 1 - 2\tan^{-1} x.$$

Equation reducible to first order – type 2

- A second order equation with no explicit x dependence,

  $$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0,$$

  can also be reduced to a first order equation, this time by changing both the dependent and the independent variables.

- Put $v = \frac{dy}{dx}$ but consider $v = v(y)$.

- Using the chain rule:

  $$\frac{d^2y}{dx^2} = \frac{dv(y)}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

- Hence we have:

  $$f\left(v\frac{dv}{dy}, v, y\right) = 0.$$

- Example:
- Solve the equation $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2.$

- Change variable: $\frac{dy}{dx} = v(y).$

- We then have:

  $$yv\frac{dv}{dy} = v^2 \Rightarrow y\frac{dv}{dy} = v$$

  and $\frac{dv}{v} = \frac{dy}{y}$ or $v = Ay.$

Equation reducible to first order – type 2

- Substituting for v, we get another separable equation:

$$\frac{dy}{dx} = Ay$$

$$\text{or } \frac{dy}{y} = A\,dx$$

$$\ln y = Ax + B$$

$$y = e^{Ax+B}$$

$$= e^{Ax}e^{B}$$

$$= Ce^{Ax}.$$

General second order linear DEs

- The general second order linear differential equation has the form:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x).$$

- Note, here we are not assuming the coefficients are constant!
- The general equation is inhomogeneous...
- ...but if f(x) = 0, the equation is homogeneous:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = 0.$$

General second order linear differential equations

- For "reasonable" coefficient functions, the homogeneous equation has the general solution:
  $y_h(x) = C_1 y_1(x) + C_2 y_2(x)$
- Here, $y_1(x)$ and $y_2(x)$ must be independent.
- If one solution, $y_1(x)$, of the homogeneous linear second order DE is known, a second independent solution, and hence the general solution, can be found.
- Do this by substituting $y_h = v(x)\, y_1(x)$ into the homogeneous equation.
- This gives a first order separable equation for v'.

- Example:
- Show that $y'' + 4y' + 4y = 0$ has a solution $y_1 = e^{-2x}$ and find the general solution of this equation, $y_h(x)$.
- Have $y_1' = -2e^{-2x}$ and $y_1'' = 4e^{-2x}$
- Hence:
  $$y'' + 4y' + 4y = 4e^{-2x} - 8e^{-2x} + 4e^{-2x} = 0$$
- So $y_1(x)$ is a solution of the DE.
- Now try $y_h = v(x)\, y_1(x)$ as general solution.
- $$\begin{aligned} y_h &= e^{-2x} v(x) \\ y_h' &= -2e^{-2x} v + e^{-2x} v' \\ y_h'' &= 4e^{-2x} v - 2e^{-2x} v' - 2e^{-2x} v' + e^{-2x} v'' \\ &= 4e^{-2x} - 4e^{-2x} v' + e^{-2x} v''. \end{aligned}$$

General second order linear differential equations

- Substitute these into the DE:

$$0 = y'' + 4y' + 4y$$
$$= 4e^{-2x}v - 4e^{-2x}v' + e^{-2x}v''$$
$$+ 4\left(-2e^{-2x}v + e^{-2x}v'\right)$$
$$+ 4e^{-2x}v$$
$$= e^{-2x}v''.$$

- So $y_h = y_1 v$ is a general solution of the DE if:

$$e^{-2x}v'' = 0$$
$$\Rightarrow v'' = 0$$
$$\text{or } v = C_1 + C_2 x.$$

- Hence the required general solution of the homogeneous equation is:

$$y_h = y_1 v$$
$$= C_1 e^{-2x} + C_2 x e^{-2x}.$$

- The general solution of an inhomogeneous equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x)$$

can be found using the above ideas if both one of the solutions of the homogeneous equation, $y_1(x)$, and a particular solution, y_p, can be deduced.

- Then we have:

$$y(x) = y_h(x) + y_p(x).$$