Differential equations

- In this lecture we will:
  - Introduce differential equations.
  - Look at how differential equations can be classified.
  - Learn how to find solutions to ordinary first order differential equations.

- Some comprehension questions for this lecture.
  - What is the order of the following equation:
    $$7\frac{d^2y}{dx^2}+2y+3=0$$
  - Is it linear? Is it homogeneous?
  - The number of radioactive decays per unit time in a sample is proportional to the number, N, of nuclei that could potentially decay, i.e. it obeys the equation:
    $$\frac{dN}{dt}=-\lambda N.$$
    Solve this equation.

Differential equations

- We will work in 1D in this section!
- "Normal" equations relate an independent variable (often labelled x) to a dependent variable (often y).
- E.g. $y = \cos x - {}^1/_3$.
- Solutions of these equations (e.g. for y = 0) are typically a number, or a collection of numbers.
- Ordinary differential equations (ODEs) relate an independent variable (x) to a dependent variable (y) and one or more of its derivatives with respect to the independent variable.
- Example $\frac{d^3y}{dx^3} + \sin x \frac{dy}{dx} = 5x^2$.
- The solution of a differential equation is typically one or more functions which relate the dependent variable to the independent variable.
- DEs are the natural way of representing many physical laws and effects, e.g. Newton's second law, Maxwell's equations (partial derivatives, 3D!), radioactivity, etc.
- The order of a differential equation is the highest derivative that appears in the equation.
- E.g. $\frac{dy}{dx} - y = 0$ is 1^{st} order.

  $x^2 \frac{d^5y}{dx^5} - \sin x \frac{dy}{dx} = 0$ is 5^{th} order.

Differential equations

- ODEs are <u>linear</u> if the dependent variable (y) and its derivatives appear to the power 1.
- E.g. $\frac{dy}{dx} = ky + x^3$ is linear.

  $L\frac{d^2y}{dx^2} + g\sin y = 0$ is non-linear.

- ODEs are <u>homogeneous</u> if each term contains either the dependent variable (y) or one of its derivatives.
- E.g. $\frac{dy}{dx} = Ay$ is homogeneous.

  $A\frac{dy}{dx} + By + C = 0$ is inhomogeneous.

- Classify these equations:
- $\frac{dy}{dx} = ky + x^2$
- $\frac{d^2u}{dt^2} + \omega^2 t = 0$
- $\frac{dx}{dt} = x^2 + 1$
- $\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 0$
- $\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 17u$
- $\frac{d^2u}{dx^2} - x\left(\frac{du}{dx}\right)^2 + u = 17u - 32$

Solving first order differential equations

- Most obvious is the solution to equations of the form:

$$\frac{dy}{dx} = f(x)$$

- All we need to do is take the "anti-derivative" (integral) of both sides:

$$\int \frac{dy}{dx} dx = \int f(x) dx$$

$$\Rightarrow y = \int f(x) dx$$

- Note, this DE may also be written...

$$dy = f(x) dx$$

- ...and its solution:

$$\int dy = \int f(x) dx$$

$$\Rightarrow y = \int f(x) dx$$

- This works as long as we can integrate f(x).
- There are many functions for which there is no exact and closed form for the integral!
- Also easy to solve are equations like:

$$\frac{dy}{dx} = g(y) \text{ or } dy = g(y) dx$$

$$\Rightarrow \int dx = \int \frac{1}{g(y)} dy$$

Solving first order DEs

- An example:

$$\frac{dy}{dx} = 6x^2 - 2x + 5$$

$$y = \int 6x^2 - 2x + 5\,dx$$

$$= 2x^3 - x^2 + 5x + c$$

- Note constant of integration, c!
- Another example:

$$\frac{dy}{dx} = y \Rightarrow dy = y\,dx$$

$$\text{and } \int \frac{dy}{y} = \int dx$$

- Integrate and add constant:

$$\ln y = x + c \text{ or } y = \exp(x + c).$$

$$\text{Hence } y = \exp(x)\exp(c) = A\exp(x)$$

Solving first order DEs – separating variables

- Equations of the form...

$$\frac{dy}{dx} = f(x)g(y)$$

- ...can be solved by separating the variables:

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

- Equation might need simplification to show that it is separable.

Solving first order DEs – separating variables

- An example:

$$\frac{dy}{dx}=\frac{y^2+xy^2}{x^2y-x^2}$$

$$=\frac{y^2(1+x)}{x^2(y-1)}$$

$$=\frac{1+x}{x^2}\frac{y^2}{(y-1)}$$

- Hence:

$$\int\frac{y-1}{y^2}\,dy=\int\frac{1+x}{x^2}\,dx$$

$$\int\frac{1}{y}-y^{-2}\,dy=\int x^{-2}+\frac{1}{x}\,dx$$

$$\ln y+\frac{1}{y}=-\frac{1}{x}+\ln x+c$$

Solving first order DEs – substitution

- All equations of the form...

$$\frac{dy}{dx}=f(x,y)$$

- ...in which f(x, y) is a homogeneous function of degree zero, that is f(tx, ty) = f(x, y), can be solved by first substituting y(x) = u(x) × x.
- We then have (differentiate product):

$$\frac{dy}{dx}=\frac{d(ux)}{dx}=x\frac{du}{dx}+u$$

- Substitute in the above equation:

$$x\frac{du}{dx}+u=f(x,ux)$$

Solving first order DEs – substitution

- Now set t = 1/x:

$$x\frac{du}{dx}+u=f\left(\frac{1}{x}x,\frac{1}{x}ux\right)=f(1,u)$$

- Can separate the variables to solve:

$$x\frac{du}{dx}+u=f(1,u)$$

$$\frac{du}{dx}=\frac{f(1,u)-u}{x}$$

$$\int\frac{du}{f(1,u)-u}=\int\frac{dx}{x}.$$

- An example: $\frac{dy}{dx}=\frac{x+3y}{2x}$

- Substitute y = ux:

$$x\frac{du}{dx}+u=\frac{x+3ux}{2x}$$

$$\Rightarrow x\frac{du}{dx}=\frac{x+3ux}{2x}-u=\frac{1}{2}+\frac{u}{2}$$

$$\Rightarrow\int\frac{2}{1+u}du=\int\frac{dx}{x}$$

$$\Rightarrow 2\ln(1+u)=\ln x+c.$$

Using $u=\frac{y}{x}$

$$2\ln\left(1+\frac{y}{x}\right)=\ln x+c$$

Solving first order DEs – integrating factor

- A further method is using an integrating factor.
- Applies to equations of the form

$$\frac{dy}{dx} + yP(x) = Q(x)$$

- The integrating factor (IF) is

$$\exp\left(\int P(x)\,dx\right).$$

- Multiply equation by IF:

$$e^{\int P(x)\,dx}\left(\frac{dy}{dx} + yP(x)\right) = e^{\int P(x)\,dx}\,Q(x).$$

- Expanding:

$$e^{\int P\,dx}\frac{dy}{dx} + e^{\int P\,dx}\,yP(x) = e^{\int P\,dx}\,Q(x)$$

- Hence (integrating w.r.t. x):

$$\int e^{\int P\,dx}\,dy + \int e^{\int P\,dx}\,yP\,dx = \int e^{\int P\,dx}\,Q\,dx$$

- Integrate first term by parts, i.e. use $\int u\,dv = uv - \int v\,du.$

$$\int e^{\int P\,dx}\,dy = e^{\int P\,dx}\,y - \int y\,d\left(e^{\int P\,dx}\right)$$

$$= y\,e^{\int P\,dx} - \int yP\,e^{\int P\,dx}\,dx$$

- Inserting back into top equation:

$$y\,e^{\int P\,dx} - \int yP\,e^{\int P\,dx}\,dx + \int e^{\int P\,dx}\,yP\,dx = \int e^{\int P\,dx}\,Q\,dx$$

Solving first order DEs – integrating factor

- Hence we have:

$$y e^{\int P\,dx} = \int e^{\int P\,dx} Q\,dx$$

$$\Rightarrow y = e^{-\int P\,dx} \int e^{\int P\,dx} Q\,dx$$

- An example:

$$\frac{dy}{dx} + 5y = e^{2x}$$

- IF is $e^{\int 5\,dx} = e^{5x}$.
- Multiply through by IF:

$$e^{5x}\frac{dy}{dx} + 5ye^{5x} = e^{2x}e^{5x}$$

- The LHS is:

$$\frac{d}{dx} ye^{5x} = 5ye^{5x} + e^{5x}\frac{dy}{dx}$$

- So we see that we can solve this equation using

$$\frac{d}{dx} y e^{5x} = e^{2x}e^{5x}$$

$$ye^{5x} = \int e^{7x}dx$$

$$= \frac{e^{7x}}{7} + c$$

- Hence:

$$y = \frac{e^{7x-5x}}{7} + ce^{-5x} = \frac{e^{2x}}{7} + ce^{-5x}$$

Integrating factors – alternative derivation

- If have equation
  $$a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x)$$
  such that
  $$\frac{d}{dx}a_1 y = a_1\frac{dy}{dx} + y\frac{da_1}{dx}$$
  $$= a_1\frac{dy}{dx} + a_0 y$$
- That is, if $\frac{da_1}{dx} = a_0.$
- Then can rewrite and easily solve:
  $$\frac{d}{dx}a_1(x)\,y = b(x) \Rightarrow a_1(x)y = \int b(x)\,dx$$
  and $y = \frac{1}{a_1(x)}\int b(x)\,dx.$
- To solve more general equations
  $$\frac{dy}{dx} + P(x)\,y = Q(x).$$
- Look for an integrating factor I(x) which equation can be multiplied by...
  $$I(x)\frac{dy}{dx} + I(x)P(x)\,y = I(x)Q(x)$$
- Choose this so that:
  $$\frac{d}{dx}I(x) = I(x)P(x)\,y$$
- To see why, cf. condition
  $$\frac{d}{dx}a_1(x) = a_0(x)$$

Integrating factors – alternative derivation

- But this is a separable equation:

$$\frac{dI(x)}{I(x)} = P(x)\,dx$$

$$\int \frac{dI(x)}{I(x)} = \int P(x)\,dx$$

$$\ln\left(I(x)\right) = \int P(x)\,dx$$

$$I(x) = \exp\left(\int P(x)\,dx\right)$$

- Our DE now becomes

$$\frac{d}{dx} I(x)\,y = I(x)Q(x)$$

- Which can be integrated to give:

$$I(x)\,y = \int I(x)Q(x)\,dx$$

$$\Rightarrow y = \frac{1}{I(x)} \int I(x)Q(x)\,dx$$

$$\text{or }\; y = e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x)\,dx$$