Vector calculus

- In this lecture we will:
  - Define the Laplace operator, or Laplacian.
  - Introduce the Poisson and Laplace equations.
  - Look at spherical polar and cylindrical coordinate systems.
- Some comprehension questions for this lecture.
  - Write down the Laplace equation.
  - Show that the surface area of a sphere of radius R is $4\pi R^2$.
  - Write down the equations that give the cylindrical coordinates r, ϕ and z in terms of the Cartesian coordinates x, y and z.

The Laplace operator and Poisson's Equation

- The Laplace operator, or the Laplacian, is the operator "divergence of gradient".
- Written ∇^2 or sometimes $\Box$.
- $\nabla^2 = \nabla \cdot \nabla$

$$= \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- E.g. $\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

- The Poisson Equation is:

  $\nabla^2 \phi(x, y, z) = g(x, y, z)$
- Setting $g(x, y, z) = 0$ in the Poisson Equation gives Laplace's Equation:

  $\nabla^2 \phi(x, y, z) = 0.$
- These equations appear often in physics.
- For example, we know:

$$\vec{E} = -\nabla \phi \text{ and } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}.$$

- Putting these together:

$$\nabla \cdot (-\nabla \phi) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

Numerical solution of Poisson's Equation

- Taylor's expansion at x_i in 1D:

$$\phi(x_i + h) \approx \phi_i + h\frac{\partial\phi_i}{\partial x} + \frac{h^2}{2}\frac{\partial^2\phi_i}{\partial x^2},$$

$$\phi(x_i - h) \approx \phi_i - h\frac{\partial\phi_i}{\partial x} + \frac{h^2}{2}\frac{\partial^2\phi_i}{\partial x^2}.$$

- Adding these gives:

$$\phi_{i+1} + \phi_{i-1} \approx 2\phi_i + h^2\frac{\partial^2\phi_i}{\partial x^2}.$$

- Hence:

$$\frac{\partial^2\phi_i}{\partial x^2} = \frac{1}{h^2}\left(\phi_{i-1} + \phi_{i+1} - 2\phi_i\right)$$

- Substitute in Poisson's equation:

$$\frac{1}{h^2}\left(\phi_{i-1} + \phi_{i+1} - 2\phi_i\right) = -\frac{\rho_i}{\varepsilon_i\varepsilon_0}$$

- Rewriting:

$$\phi_i^{new} = \frac{h^2}{2}\left(\frac{\phi_{i+1} + \phi_{i-1}}{h^2} + \frac{\rho_{i,j}}{\varepsilon_{i,j}\varepsilon_0}\right)$$

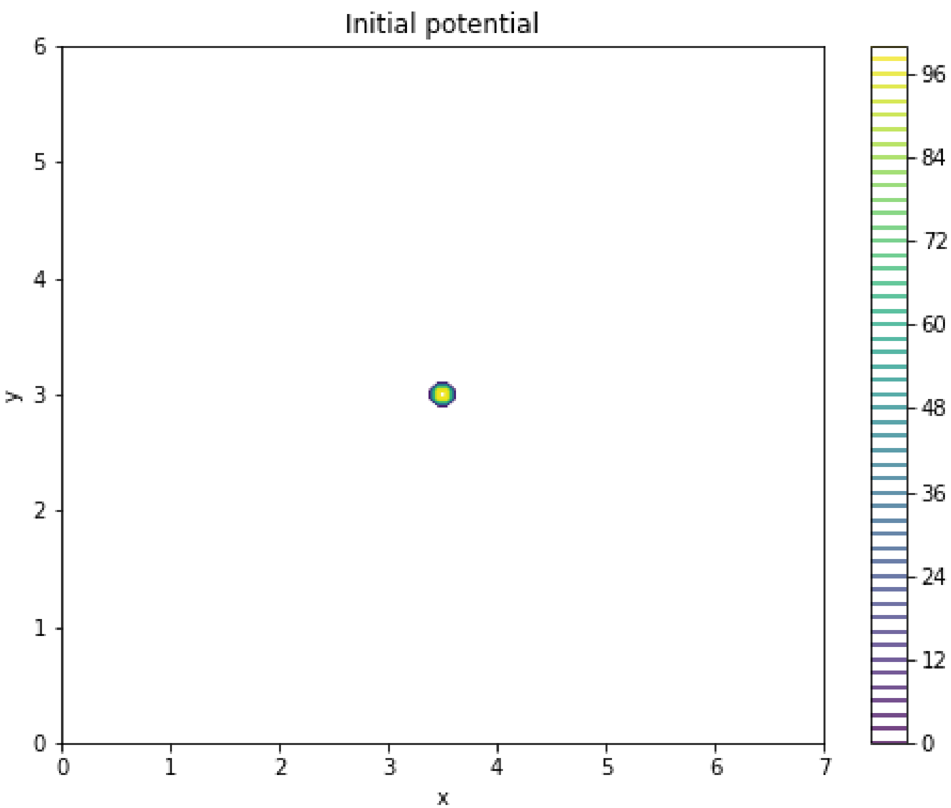
- Extending to 2D:

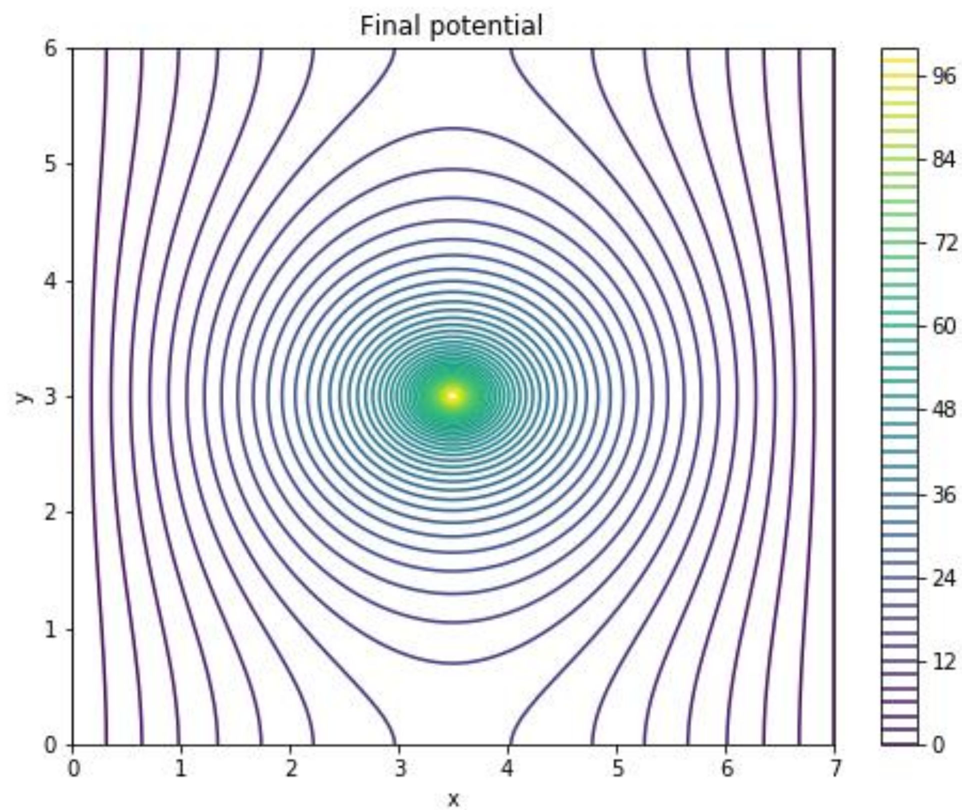
$$\phi_{i,j}^{new} = \frac{h_x^2 h_y^2}{2(h_x^2 + h_y^2)} \times \left(\frac{\phi_{i-1,j} + \phi_{i+1,j}}{h_x^2} + \frac{\phi_{i,j-1} + \phi_{i,j+1}}{h_y^2} + \frac{\rho_{i.j}}{\varepsilon_{i,j}\,\varepsilon_0}\right).$$

- Use this to solve iteratively for ϕ.
- "Tortoise convergence" i.e. sure, but slow!
- Look at an example…

Numerical solution of Poisson’s Equation

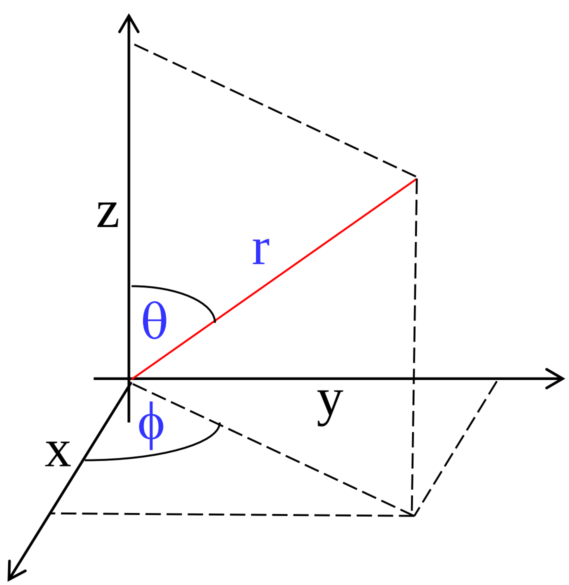
- Put a charged blob in the centre of a box with side walls at earth potential.
- Use the method of relaxation to calculate the resulting potential distribution.





Spherical polar coordinates

- Sometimes use coordinate systems other than Cartesian (x, y) or (x, y, z).
- E.g. circular motion, use (r, θ) rather than (x, y) coordinates.
- Consider spherical polar coords:



- Relationship between Cartesian and spherical polar coordinates:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

- Note, these are "physics" definitions, mathematicians often label the θ and ϕ coordinates the other way round!
- Inverting the above:

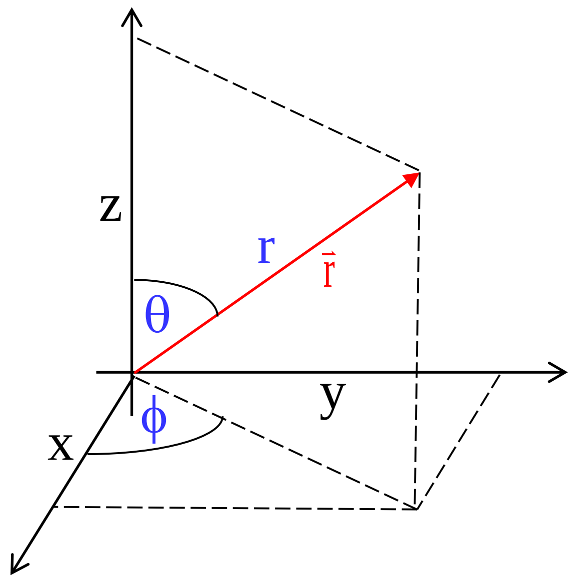
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \mathrm{acos}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\phi = \mathrm{atan}\left(\frac{y}{x}\right)$$

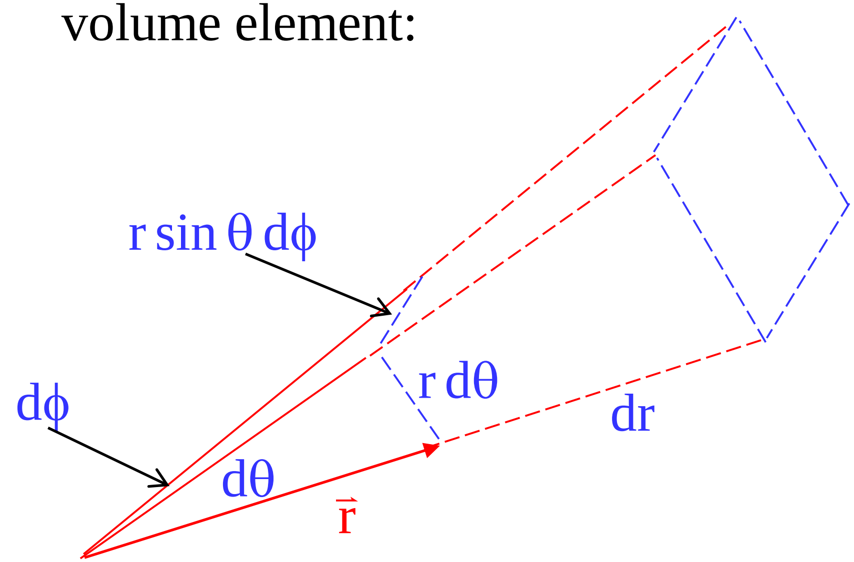
Spherical polar coordinates

- Line element from $\vec{r} = (r, \theta, \phi)$ to $\vec{r} + d\vec{r}$.



- $d\vec{r} = (dr, r\,d\theta, r\sin\theta\,d\phi)$

- Variation of the spherical polar coordinates produces the following volume element:



- Volume of this element is:

$$dV = dr \times r\,d\theta \times r\sin\theta\,d\phi$$

$$= r^2 \sin\theta\,d\theta\,d\phi\,dr$$

Spherical polar coordinates

- The surface element spanning from θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$ is

$$dS = r\,d\theta \times r\sin\theta\,d\phi$$
$$= r^2 \sin\theta\,d\theta\,d\phi$$

- Solid angle subtended by this element

$$d\Omega = \frac{dS}{r^2}$$
$$= \sin\theta\,d\theta\,d\phi$$

- Can calculate area of sphere of radius R by integrating over θ and ϕ (try it!):

$$A = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta\,d\theta\,d\phi$$

- Get volume of sphere by integrating over r, θ and ϕ.

$$V = \int_0^R\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\,d\theta\,d\phi\,dr$$
$$= \frac{R^3}{3}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi$$
$$= \frac{R^3}{3}\int_0^{2\pi} -\cos\theta\Big|_0^{\pi}\,d\phi$$
$$= \frac{2R^3}{3}\int_0^{2\pi} d\phi$$
$$= \frac{4\pi R^3}{3}$$

Spherical polar and cylindrical coordinates

- Gradient in Spherical Polar coordinate system:

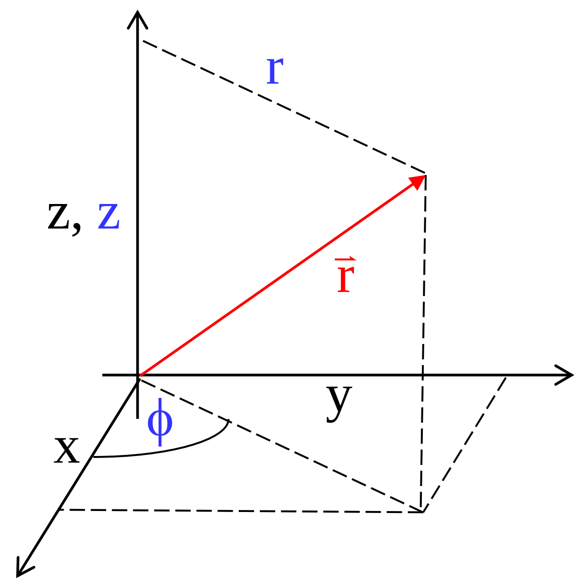
$$\nabla V = \left(\frac{\partial V}{\partial r} \quad \frac{1}{r}\frac{\partial V}{\partial \theta} \quad \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \right)$$

$$= \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

- Divergence:

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\sin\theta A_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi$$

- Expressions for curl and Laplacian in Spherical Polars are messy – look them up when you need them!
- Cylindrical coordinate system also often useful.



$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = z$$

Cylindrical Coordinates

- Gradient in cylindrical coordinate system:

$$\nabla V = \left(\frac{\partial V}{\partial r} \quad \frac{1}{r}\frac{\partial V}{\partial \phi} \quad \frac{\partial V}{\partial z} \right)$$

$$= \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

- Divergence:

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r} rA_r + \frac{1}{r}\frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z$$

- Cartesian, spherical polar and cylindrical coordinates are the most commonly used systems.
- General approach to use of orthogonal curvilinear coordinate systems described in text book.
- Good introduction to some of the ideas that are important in General Relativity.