Vector calculus – some odds and ends

- In this lecture we will:
  - Look again at finding the potential associated with a field using the expression:

$$\phi = \int_C \vec{E} \cdot d\vec{r}.$$

  - Look at another way of finding the potential associated with a field.
  - Look at an exam question or two.

- Some comprehension questions for this lecture.
  - Calculate the curl of the field

$$\vec{E} = \begin{pmatrix} 6xy^2 + 2xz^3 \\ 6x^2y - 6y^2z \\ 3x^2z^2 - 2y^3 \end{pmatrix}.$$

  - Is it possible to represent this field as the gradient of a scalar potential?
  - What is the quantity for gravitational fields that is analogous to electric charge for electric fields?

More on deriving a potential from a field

- Check path independence.
- Example field $\vec{E}(x,y) = \begin{pmatrix} 2x+y & x \end{pmatrix}$.
- Find the associated potential,

$$\phi = \int_C \vec{E}\cdot d\vec{r}$$

- Integrate along $\vec{r}(t) = \begin{pmatrix} x(t) & y(t) \end{pmatrix}$ with t running from 0 to 1.
- Choose $\vec{r}(t) = \begin{pmatrix} xt^2 & yt^2 \end{pmatrix}$ so:

$$\phi(x,y) = \int_0^1 \vec{E}\left(x(t), y(t)\right)\cdot \frac{d\vec{r}(t)}{dt}\,dt$$

$$= \int_0^1 E_x\left(x(t), y(t)\right)\frac{dx(t)}{dt}\,dt + \int_0^1 E_y\left(x(t), y(t)\right)\frac{dy(t)}{dt}\,dt$$

- $\dfrac{d\,x(t)}{dt} = \dfrac{d\,xt^2}{dt} = 2xt,\ \dfrac{d\,y(t)}{dt} = 2yt.$
- This then gives:

$$\phi = \int_0^1 [2xt^2 + yt^2]\times 2xt\,dt + \int_0^1 xt^2 \times 2yt\,dt$$

$$= \int_0^1 [4x^2 + 2xy]\times t^3\,dt + \int_0^1 2xy\times t^3\,dt$$

$$= \frac{t^4}{4}\times(4x^2+2xy) + \frac{t^4}{4}\times 2xy\Big|_0^1$$

$$= x^2 + \frac{xy}{2} + \frac{xy}{2}$$

$$= x^2 + xy.$$

- Same result as with $\vec{r}(t) = \begin{pmatrix} xt & yt \end{pmatrix}$.

Alternative way of getting a potential from a field

- See by doing an example.
- $\vec{E} = \begin{pmatrix} 6xy^2 + 2xz^3 \\ 6x^2y - 6y^2z \\ 3x^2z^2 - 2y^3 \end{pmatrix}.$
- Find ϕ such that: $\nabla\phi = \begin{pmatrix} 6xy^2 + 2xz^3 \\ 6x^2y - 6y^2z \\ 3x^2z^2 - 2y^3 \end{pmatrix}.$
- That is $\frac{\partial}{\partial x}\phi = 6xy^2 + 2xz^3$ [1]

  $\frac{\partial}{\partial y}\phi = 6x^2y - 6y^2z$ [2]

  $\frac{\partial}{\partial z}\phi = 3x^2z^2 - 2y^3$ [3]

- From [1], integrating w.r.t. x:

  $\phi = 3x^2y^2 + x^2z^3 + f(y,z).$
- Take partial derivative w.r.t. y.

  $$\frac{\partial}{\partial y}\phi = 6x^2y + \frac{\partial}{\partial y}f(y,z).$$
- From [2]:

  $$6x^2y + \frac{\partial}{\partial y}f(y,z) = 6x^2y - 6y^2z$$

  $$\Rightarrow \frac{\partial}{\partial y}f(y,z) = -6y^2z$$

  $$\Rightarrow f(y,z) = -2y^3z + g(z).$$
- This now gives:

  $$\phi = 3x^2y^2 + x^2z^3 - 2y^3z + g(z).$$

Potential from a field

- Now take the partial derivative with respect to z:

$$\frac{\partial}{\partial z}\phi = 3x^2z^2 - 2y^3 + \frac{\partial}{\partial z}g(z).$$

- Compare this to [3]:

$$3x^2z^2 - 2y^3 = 3x^2z^2 - 2y^3 + \frac{\partial}{\partial z}g(z)$$

$$\Rightarrow \frac{\partial}{\partial z}g(z) = 0$$

$$\Rightarrow g(z) = \text{const.}$$

- We now have:

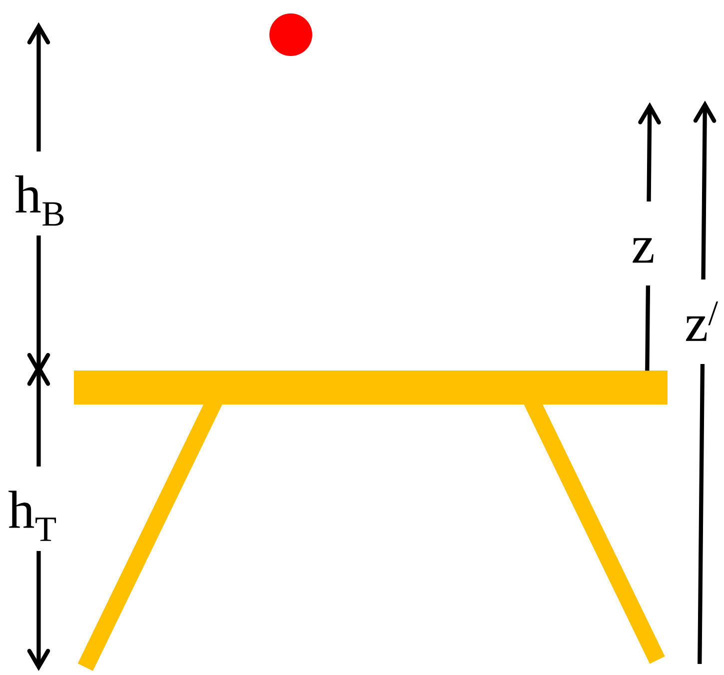
$$\phi = 3x^2y^2 - 2y^3z + x^2z^3 + \text{const.}$$

Constants in potentials

- Potentials are related to potential energies.
- Some examples:
- Electric potential.
  - (Scalar) field V(x y z).
  - Units, volts = joules/coulomb.
  - A charge q in the field V has a potential energy U = qV (joules).
- Gravitational potential.
  - G(x, y, z) = g × z (close to Earth).
  - Units J/kg.
  - A mass m in the field G has a potential energy U = m × G (joules).

Constants in potentials

- Can measure differences in potential energy (and hence potential), but not absolute values.
- Gravitational example:



- Gravitational potential in "table coordinates" is $G(z) = gz$.
- Gravitational potential in "floor coordinates" is $G(z') = gz' = gz + gh_T$.
- Potential energy change when ball falls to table, in table coordinates:
  - $\Delta U = mgh_B - 0$
    $= mgh_B$.
- Potential energy change when ball falls to table, in floor coordinates:
  - $\Delta U = mg(h_B + h_T) - mgh_T$
    $= mgh_B$.

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MATHEMATICS FOR PHYSICISTS II

TIME ALLOWED: 3 hours

INSTRUCTIONS TO CANDIDATES

Answer all questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

Answer either part (a) or part (b) of questions 2 and 3.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

The marks allotted to each part of a question are indicated in square brackets.

All symbols have their usual meanings unless otherwise stated.

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Question 1.

(a)

The matrices **A**, **B** and **C** are given by:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Calculate the products **AB** and **CA**. [6]

State which of the following expressions are correct:

(i) $\mathbf{A} = \mathbf{B}$.

(ii) $\mathbf{BA} = \mathbf{I}$, where **I** is the unit matrix.

(iii) $\mathbf{A}^{-1} = \mathbf{B}$.

(iv) $\mathbf{B} = \frac{1}{4}\begin{pmatrix} 0 & 2 & -2 \\ 1 & 2 & -2 \\ 1 & 1 & 0 \end{pmatrix}$. [4]

Calculate the determinant $|\mathbf{A}|$ and the transpose $\mathbf{A}^T$ of **A**. [5]

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(b)

Firstly, use Cramer's method to solve the system of simultaneous equations:

$$\begin{aligned} y + 2z &= -2 \\ z + 2x &= 5 \\ x + 2y &= 3. \end{aligned}$$ [6]

Secondly, write down the above simultaneous equations in the matrix form $\mathbf{A}\vec{x} = \vec{c}$, where **A** is a 3×3 matrix and $\vec{x}$ and $\vec{c}$ are column vectors. [2]

Invert **A** and use the inverted matrix to again solve the system of simultaneous equations. [7]

(c)

A vector field is defined by $\vec{E}(x,y,z) = x^2\,\hat{i} + xy\,\hat{j} + z^2\,\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the x, y and z directions of a Cartesian coordinate system.

Find the divergence $\nabla \cdot \vec{E}$. [3]

Calculate the curl, $\nabla \times \vec{E}$. [7]