Phys108 – Mathematics for Physicists II

- Lecturer:
  - Prof. Tim Greenshaw.
  - Oliver Lodge Lab, Room 333.
  - Office hours, Fri. 11:30…13:30.
  - Email green@liv.ac.uk
- Lectures:
  - Monday 14:00, HSLT.
  - Tuesday 13:00, HSLT.
  - Thursday 09:00, HSLT.
- Problems Classes:
  - Friday 9:00...11:00.
  - Central Teaching Labs, GFlex.
- Outline syllabus:
  - Matrices.
  - Vector calculus.
  - Differential equations.
  - Fourier series.
  - Fourier integrals.
- Recommended textbook:
  - “Calculus, a Complete Course”, Adams and Essex, (Pub. Pearson).
- Assessment:
  - Exam end of S2: 70%.
  - Problems Classes: 20%.
  - Homework: 10%.

Vector calculus – the gradient of a scalar field

- In this lecture we will:
  - Revise partial differentiation.
  - Introduce scalar and vector fields.
  - Look at some methods of visualising scalar and vector fields.
  - Define the gradient of a scalar field.
  - Look at electric fields and potentials.

- Some comprehension questions for this lecture.
  - Explain which of the following can be represented as scalar and which as vector fields:
    - Atmospheric pressure.
    - Ocean currents.
    - Height above sea level across the UK.
  - Calculate the electric field associated with the electric potential $\phi(x, y, z) = 4z$.

Some revision – partial derivatives

- Consider a function of two variables, f(x, y).
- The partial derivatives of this function w.r.t. x and y are defined by:

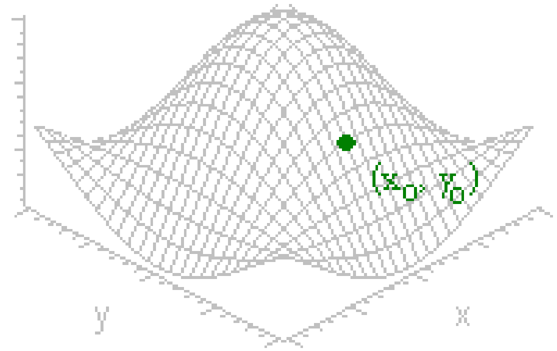
$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

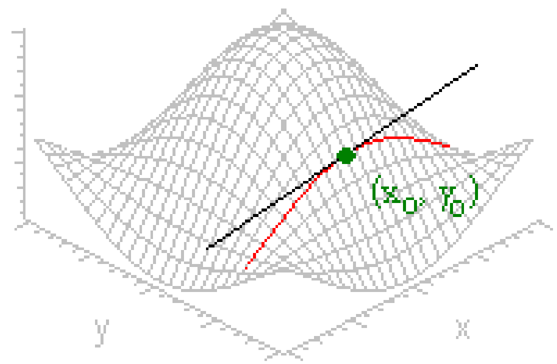
$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

- Example: $f(x, y) = xy^2$.

$$\frac{\partial f}{\partial x} = y^2, \ \frac{\partial f}{\partial y} = 2xy.$$

- Geometrically, consider z = f(x, y) as shown opposite:





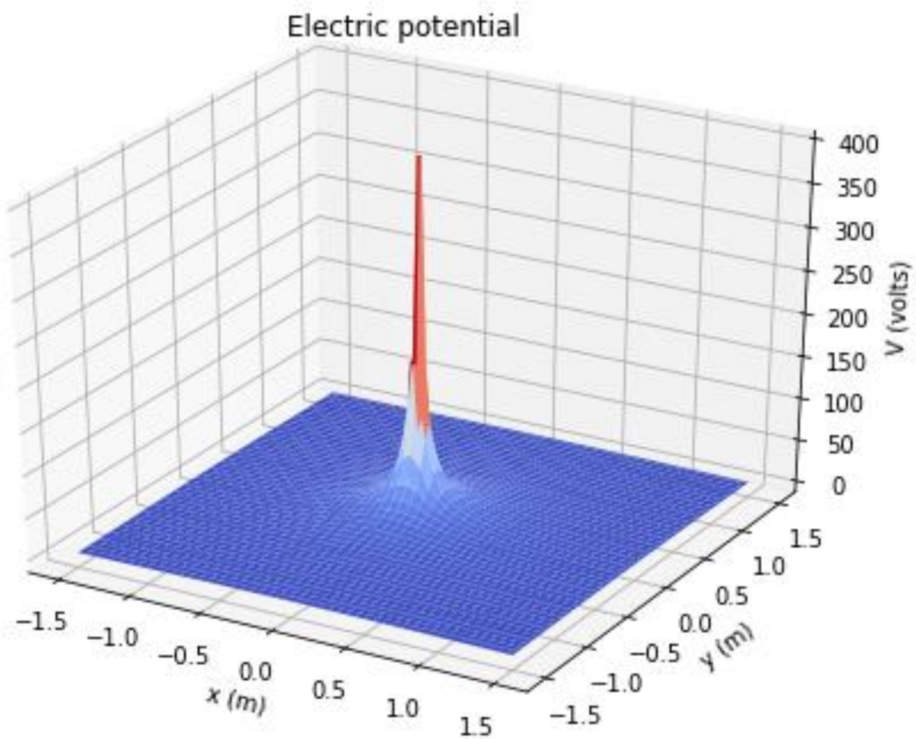
- Keep $y = y_0$, then $z = f(x, y_0)$ traces out the red curve shown.
- The slope of this curve at (x_0, y_0) is given by $\frac{\partial}{\partial x} z(x_0, y_0) = \left.\frac{\partial z}{\partial x}\right|_{x_0, y_0}$.

Some revision – partial derivatives

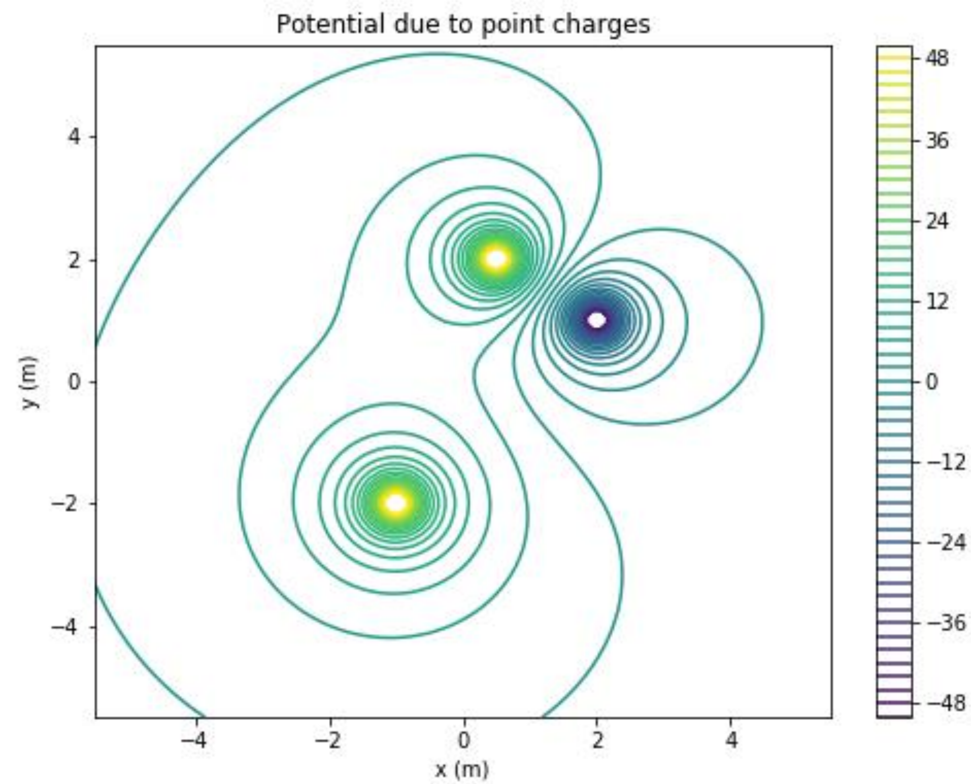
- Calculate the following derivatives:
- $\frac{\partial}{\partial x}\left(\cos 4x - \sin 3y + \exp[-2xz]\right) =$
- $\frac{\partial}{\partial z}\left(\cos 4x - \sin 3y + \exp[-2xz]\right) =$

Scalar fields

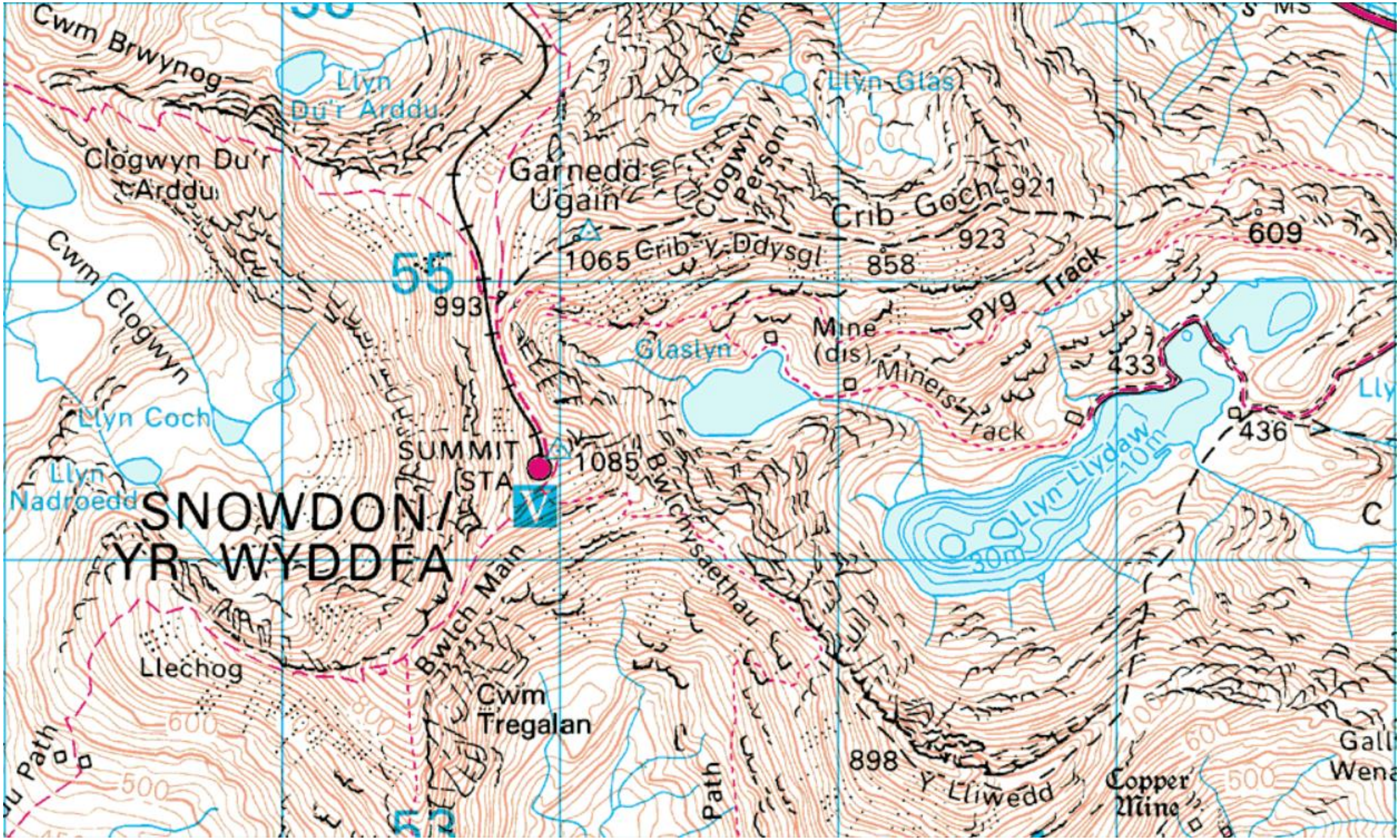
- A scalar field is a scalar that is defined at all points in space.
- Example, electric potential around point charge:



- Can plot in “3D” for field defined in (x, y) plane, or use contour plot.
- The contours are “equipotentials”:

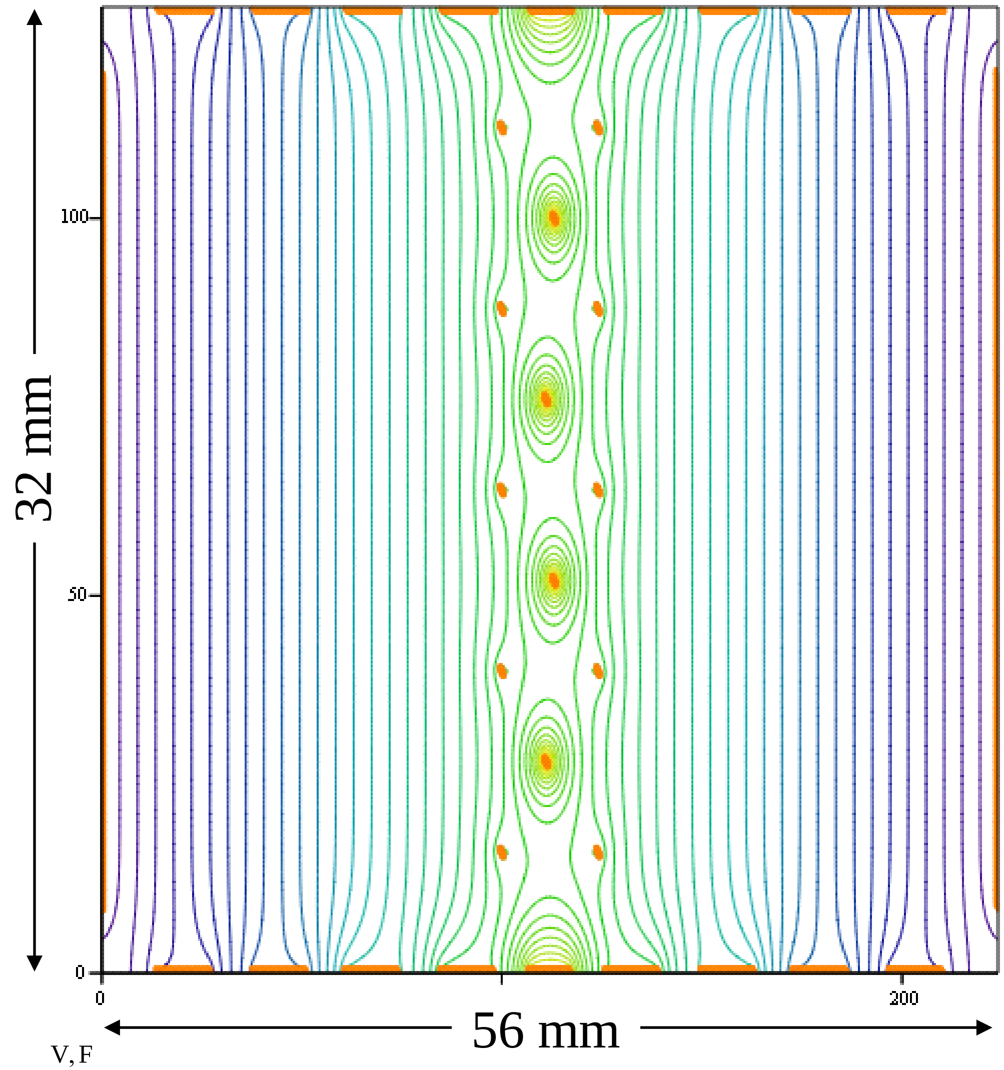


Contour plot of Snowdon



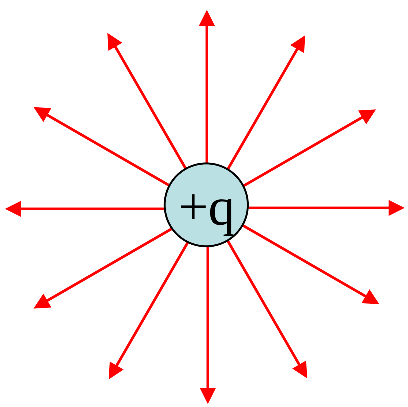
Scalar fields and equipotentials

- Electric potential in drift chamber illustrated using equipotentials.
- Electric field always normal to equipotentials.
- Electrons produced in drift volume by high energy charged particle passing through gas in chamber.
- Electrons drift along electric field lines to anode wires (central potential wells) where they produce electrical signals.
- Drift electric field ~ 1 MV/m.
- Using information on time taken for electrons to reach wires, reconstruct path of high energy charged particle.

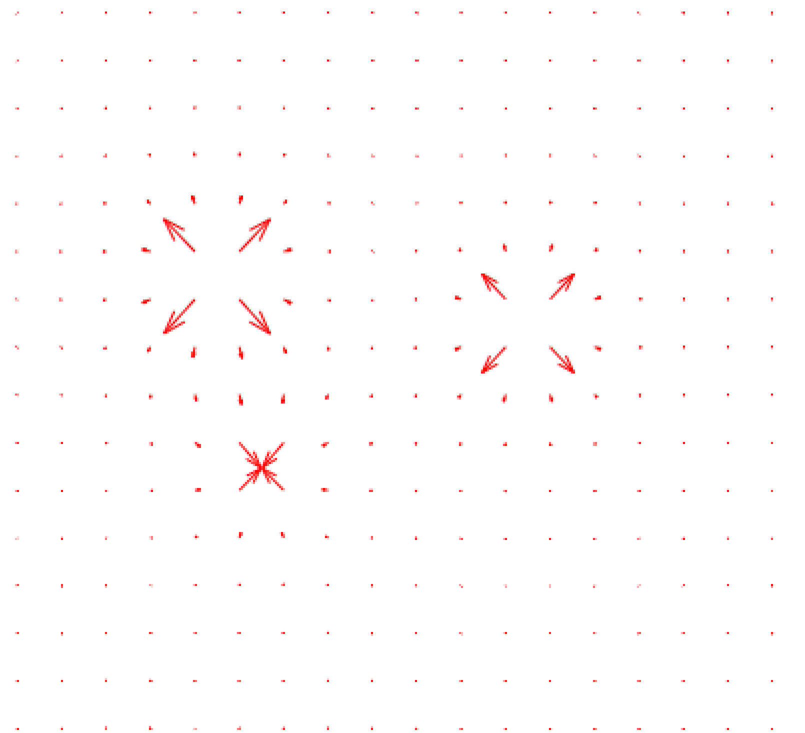


Vector fields

- A vector field is a vector that is defined at all points in space.
- Physical examples include the electric field, e.g. that surrounding a point charge can be sketched as:



- Can represent a vector field defined in the (x, y) plane using arrows in the direction of the vector whose length is proportional to the magnitude.



Scalar and vector field examples

- A scalar field is defined by:

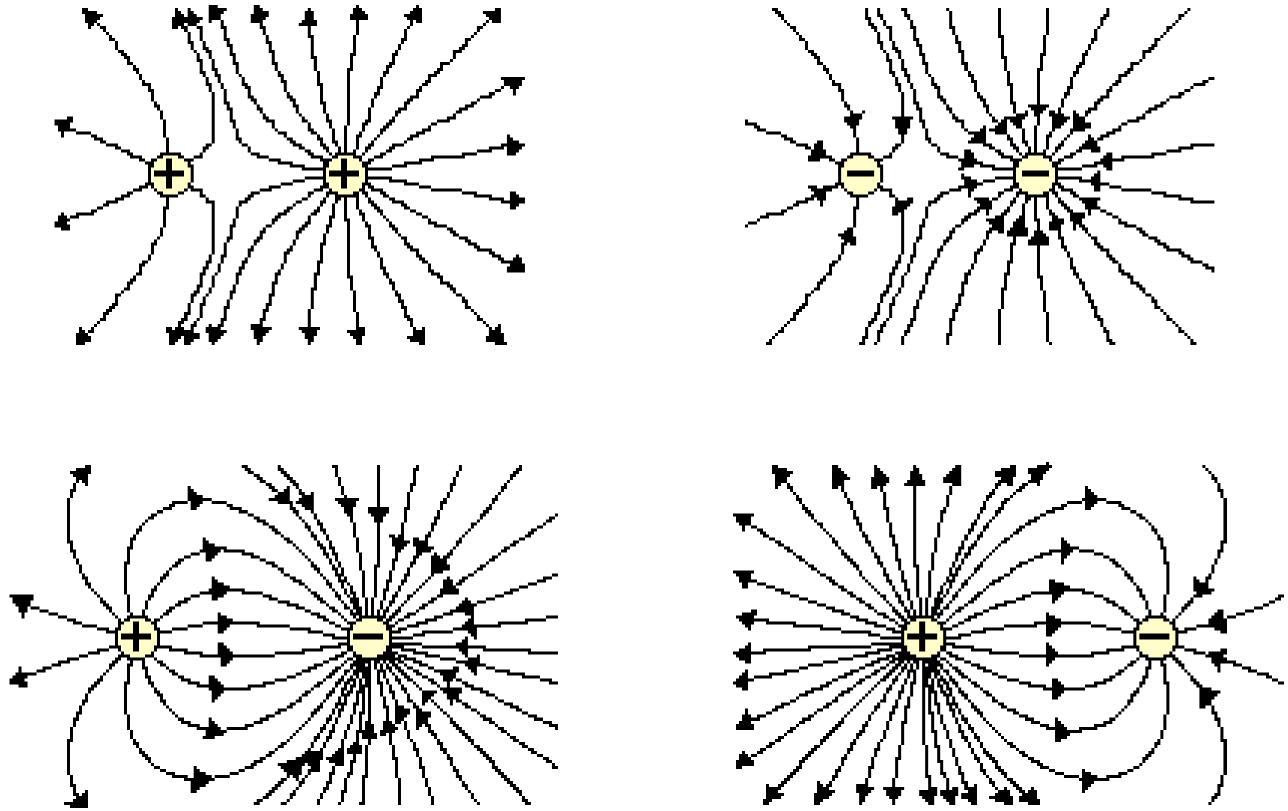
  $\phi(x, y, z) = 4x^2 - 3y + xz$
- What is the value of the field at the point $(x, y, z) = (1, 2, 2)$?

- A vector field is defined by:

  $$\vec{E}(x, y, z) = \begin{pmatrix} 1 \\ \cos x \\ \sin y \end{pmatrix}$$
- What is the magnitude of the field at the point $(x, y, z) = (0, \frac{\pi}{2}, 0)$?
- What is its direction at the origin?

Vector fields and field lines

- Electric field lines are another way of visualising E fields.
- Lines trace path followed by (slow) test charge.
- Density of lines proportional to field strength.
- Examples shown opposite.
- Note that positive and negative charges are not balanced – how can you tell this?

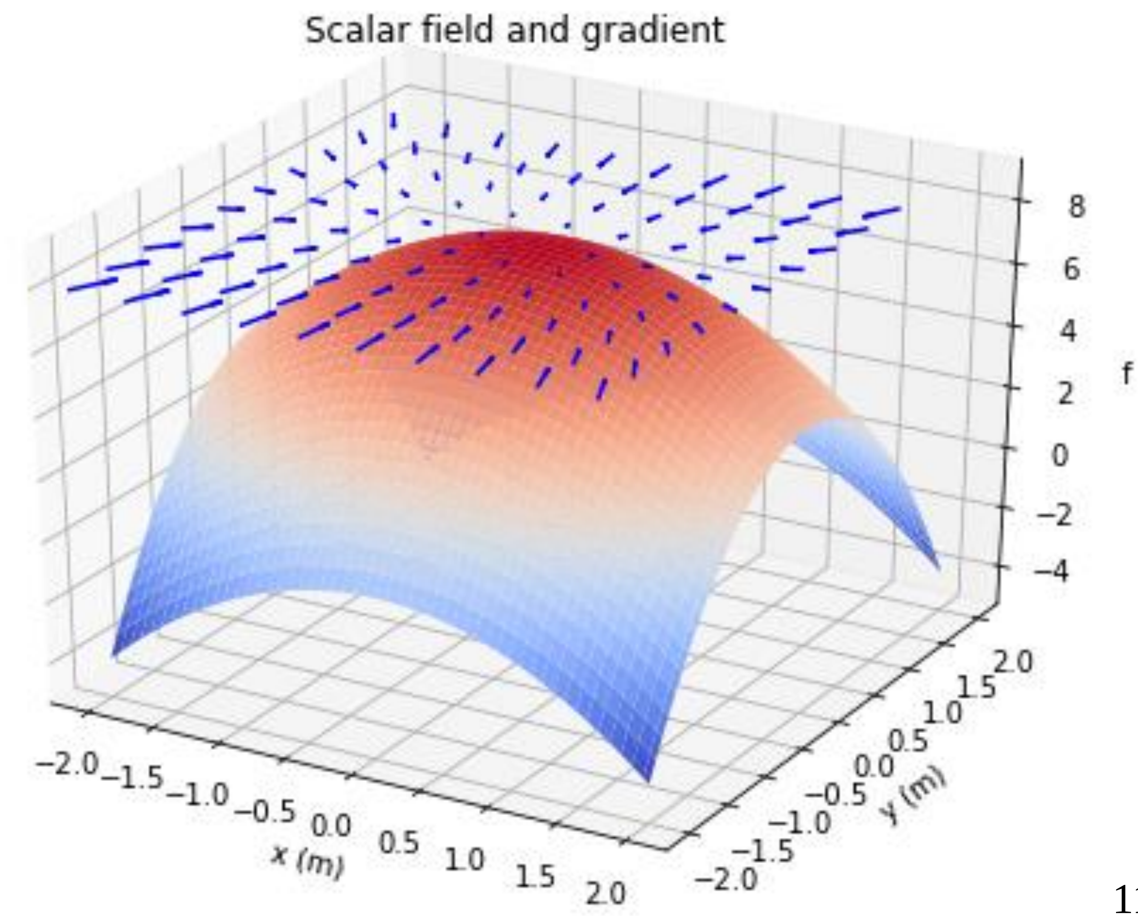


Gradient of a scalar field

- The gradient of a scalar field f(x, y, z) is defined by:
- $$\nabla f(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x} f(x,y,z) \\ \frac{\partial}{\partial y} f(x,y,z) \\ \frac{\partial}{\partial z} f(x,y,z) \end{pmatrix}$$
- The gradient of a scalar field is a vector field.
- Can also write as row vector:

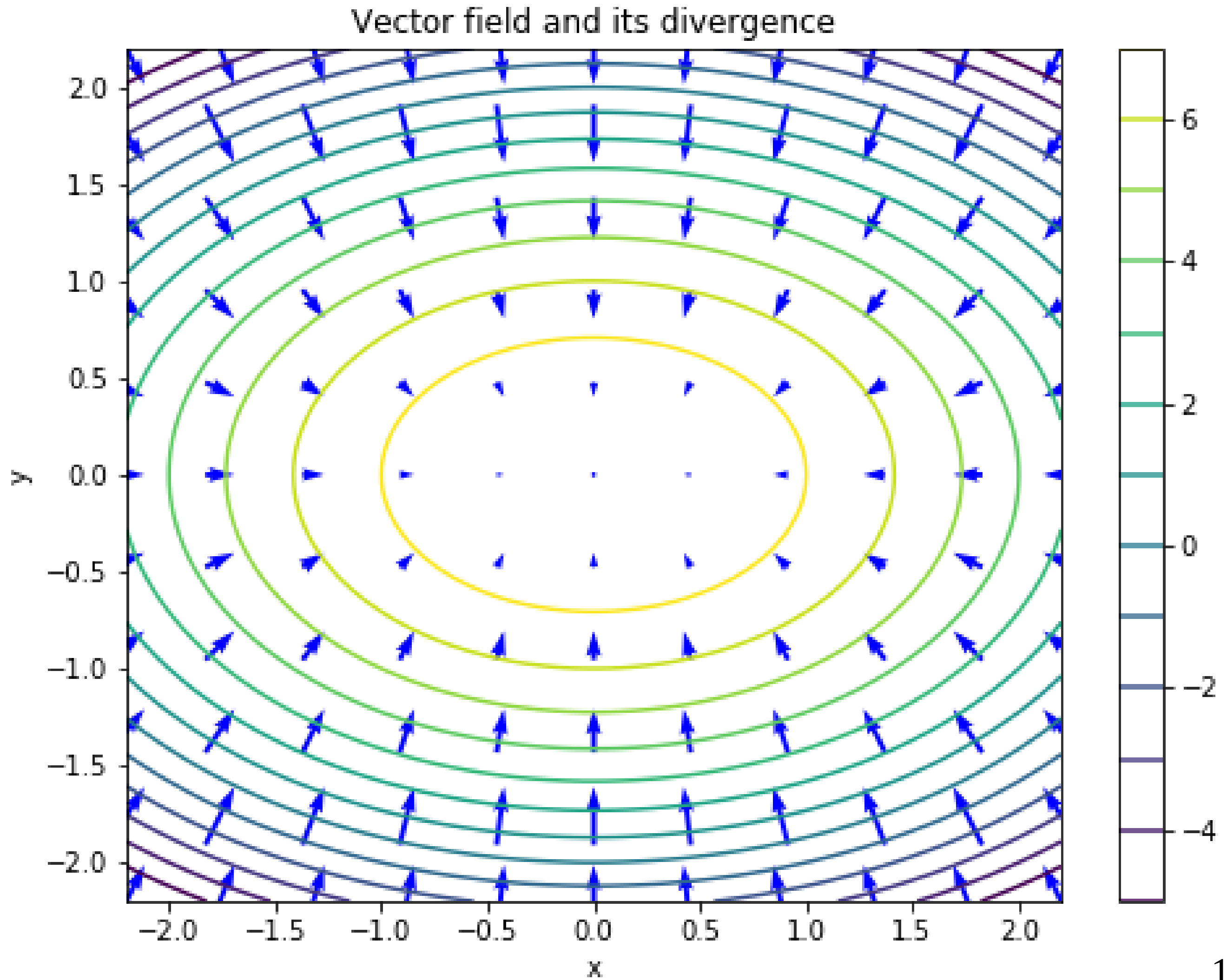
$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

- Example, in 2D (so can draw on screen).
- $f(x, y) = -x^2 - 2y^2 + 8$, $\nabla f = \begin{pmatrix} -2x & -4y \end{pmatrix}$



Gradient of a scalar field

- Plot the scalar field $f(x, y) = -x^2 - 2y^2 + 8$ as a contour plot.
- Plot the field's gradient $\nabla f = \begin{pmatrix} -2x & -4y \end{pmatrix}$ as a vector plot.



Gradient of a scalar field

- The gradient vectors point in the direction of the steepest slope of the scalar field at the positions at which they are defined.
- The magnitude of the gradient vector gives the steepness of the slope (the gradient).
- A physical example:

$$\vec{E} = -\nabla V \equiv -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

- Around a point charge q:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}}.$$

- Calculate E field using our prescription, x component:

$$\frac{\partial V}{\partial x} = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$
$$= \frac{q}{4\pi\varepsilon_0}\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\times 2x$$
$$= -\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2+z^2}}$$
$$= -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{x}{r}.$$

Calculating $\vec{E}$ from V

- Doing the same for the y and z components we have:
- $$\frac{\partial V}{\partial y} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{y}{r}$$
- and
- $$\frac{\partial V}{\partial z} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{z}{r}.$$
- Hence:
- $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right).$$
- Now, $x/r = \cos\theta_{xr}$ is the component of the radius vector in the x direction, y/r that in the y direction and z/r that in the z direction, so we see:
- $$\left|\vec{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$ and...
- ...the E field is directed radially away from the charge, as expected.