2 b)

$$m\frac{d^2z}{dt^2} + c\frac{dz}{dt} + kz = F(t)$$

$$\rightarrow \quad \frac{d^2z}{dt^2} + 5\frac{dz}{dt} + 4z = 70\cos 2t$$

Auxiliary eqn.

$$m^2 + 5m + 4 = 0$$

$$\Rightarrow (m+1)(m+4) = 0$$

$$m_1 = -1 \quad m_2 = -4$$

$$\Rightarrow z_c = Ae^{-t} + Be^{-4t}$$

$$z_p = C\sin 2t + D\cos 2t$$

$$\dot{z}_p = 2C\cos 2t - 2D\sin 2t$$

$$\ddot{z}_p = -4C\sin 2t - 4D\cos 2t$$

$$\Rightarrow -4C\sin 2t - 4D\cos 2t + 10C\cos 2t - 10D\sin 2t$$
$$+ 4C\sin 2t + 4D\cos 2t = 70\cos 2t$$

$$\Rightarrow -4C - 10D + 4C = 0 \Rightarrow D = 0$$

and $-4D + 10C + 4D = 70 \Rightarrow C = 7$

$\Rightarrow z_p = 7\sin 2t$ and $z = Ae^{-t} + Be^{-4t} + 7\sin 2t$

1 e) $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$

$P_0(x) = 1$ con.

$(1-x^2)\times 0 - 2x\times 0 + 0(0+1)\times 1 = 0$ QED

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_2'(x) = 3x \quad P_2''(x) = 3$

$(1-x^2)\times 3 - 2x\times 3x + 2\times 3\times\frac{1}{2}(3x^2-1)$

$= 3 - 3x^2 - 6x^2 + 9x^2 - 3 = 0$ QED

$f(x) = 9x^5 + 5x^3 - 6x = aP_1(x) + bP_3(x) + cP_5(x)$

$\Rightarrow ax + \frac{b}{2}(5x^3-3x) + \frac{c}{8}(63x^5 - 70x^3 + 15x)$

$= 9x^5 + 5x^3 - 6x$

$\Rightarrow ax + \frac{5b}{2}x^3 - \frac{3b}{2}x + \frac{63c}{8}x^5 - \frac{70cx^3}{8} + \frac{15cx}{8}$

$= 9x^5 + 5x^3 - 6x$

$\Rightarrow \frac{63c}{8} = 9$ or $c = \frac{8\times 9}{63} = \frac{3\times 8}{21} = \frac{8}{7}$

$\frac{5b}{2} - \frac{70}{8}c = 5 = \frac{5b}{2} - \frac{70\times 8}{8\times 7} = 5, \quad \frac{5b}{2} = 5 + 10, \quad b = 6$

$a - \frac{3b}{2} + \frac{15c}{8} = -6 \Rightarrow a - 9 + \frac{15}{7} = -6 \Rightarrow a = 3 - \frac{15}{7}$

$a = \frac{6}{7}$

$$\int_{-1}^{1} f(x) P_3(x)\,dx = \int_{-1}^{1} \left(a P_1 + b P_3 + c P_5\right) P_3\,dx$$

$$= b\,\frac{2}{2\times 3+1} = \frac{2}{7}\,b = \frac{2\times 6}{7} = \frac{12}{7}$$

$$\int_{-1}^{1} f(x)\, P_3\,dx = \int_{-1}^{1} \left(9x^5 + 5x^3 - 6x\right) \frac{1}{2}\left(5x^3 - 3x\right) dx$$

$$= \int_{-1}^{1} \frac{45}{2}x^8 - \frac{27}{2}x^6 + \frac{25}{2}x^6 - \frac{15}{2}x^4 - \frac{30}{2}x^4 + \frac{18}{2}x^2\,dx$$

$$= \int_{-1}^{1} \frac{45}{2}x^8 - x^6 - \frac{45}{2}x^4 + 9x^2\,dx$$

$$= \int_{0}^{1} 45x^8 - 2x^6 - 45x^4 + 18x^2\,dx$$

$$= \frac{45}{9} - \frac{2}{7} - \frac{45}{5} + \frac{18}{3} = 5 - \frac{2}{7} - 9 + 6$$

$$= 2 - \frac{2}{7} = \frac{12}{7}$$

So orthonormality gives same result as direct calculation.