FourierMolly

May 13, 2019

1 Fourier Examples

1.1 Introduction

Fourier series can be used to represent periodic functions as a sum of sine and cosine terms with different amplitudes and freqencies. For example, a function $f(t)$, with period T, can be written:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\tfrac{2n\pi t}{T}\right) + b_n \sin\left(\tfrac{2n\pi t}{T}\right). \tag{1}$$

The coefficients a_0, a_n and b_n may be determined as follows:

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(t)\,dt \\ a_n &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)\,dt \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt. \end{aligned} \tag{2}$$

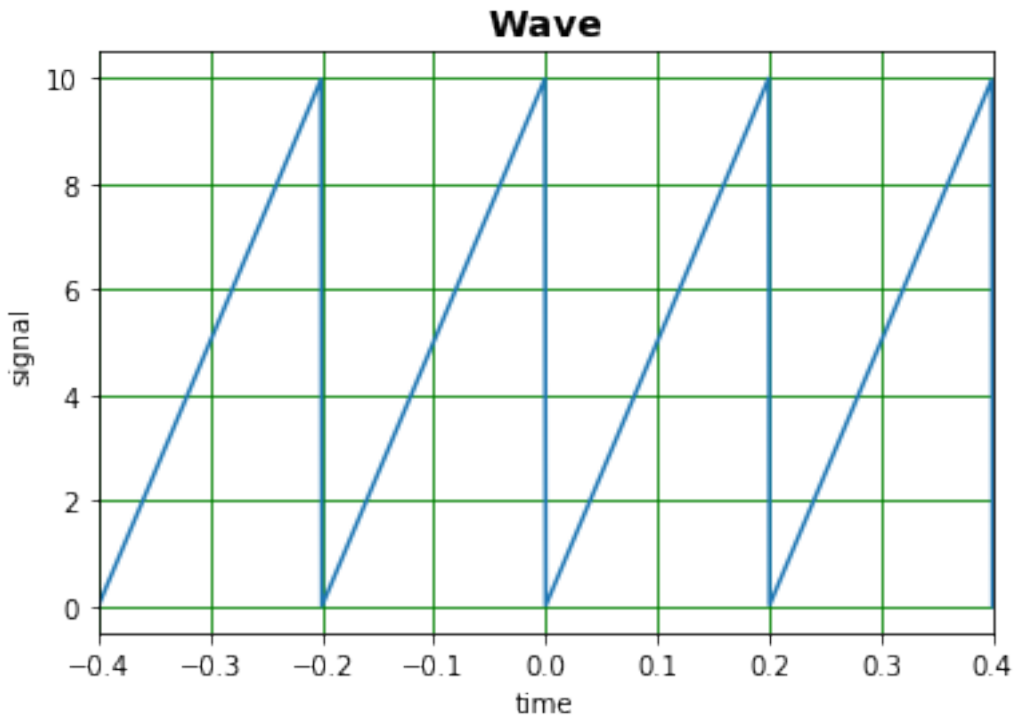
The function here is:

$$\begin{aligned} f(t) &= 50t + 10 \text{ if } -0.2 < t < 0.0, \\ &= 2 \text{ if } 0.0 < t < 0.2. \end{aligned} \tag{3}$$

We have $T = 0.4$. The graph of $f(t)$ is shown below.

[1]:
```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
%matplotlib inline
from matplotlib import cm
import sympy as sp
from IPython.display import display, Markdown, Latex
#
```

```
# define the flat triangle function
def SawTooth(t, T):
    '''
    Saw tooth function centered on zero with period T
    '''
    lArr = len(t)
    Tarr = T*np.ones(lArr)
    arg = t - ((t - T/2)//T)*T - T
    boolA = np.logical_and(-Tarr < arg, arg < 0.0)
    boolB = np.logical_and(0*Tarr < arg, arg < Tarr)
    fA = 10 + 50*arg
    fB = 50*arg
    #
    f = boolA*fA + boolB*fB
    #
    return f
#
T = 0.4
nT = 1000
tBot = -0.4
tTop = 0.4
tArr = np.linspace(tBot, tTop, nT)
fArr = np.zeros(nT)
fArr = SawTooth(tArr, T)
plt.plot(tArr, fArr)
plt.title("Wave", size = 14, weight = "bold")
plt.xlabel("time")
plt.ylabel("signal")
plt.axis([tBot, tTop, -0.5, 10.5])
plt.grid(color = 'g')
plt.savefig("SawTooth.png")
```



Calculate a_0:

[2]:
```
t = sp.Symbol("t")
a0 = 1/T*(sp.integrate(50*t + 10, (t, -T/2, 0.0)) + sp.integrate(50*t, (t, 0.0,␣
↪T/2)))
display(Markdown("$a_0 = " + sp.latex(a0) + "$"))
```

$a_0 = 5.0$

We have $a_0 = 5$.

Determine the expression for the a_n:

[3]:
```
n = sp.Symbol("n")
an = 2/T*(sp.integrate((50*t + 10)*sp.cos(2*n*sp.pi*t/T), (t, -T/2, 0.0)) +
          sp.integrate(50*t*sp.cos(2*n*sp.pi*t/T), (t, 0.0, T/2)))
display(Markdown("$a_n = " + sp.latex(sp.simplify(an)) + "$"))
```

$$a_n = \begin{cases} \frac{10.0 \sin(\pi n)}{\pi n} & \text{for } n > -\infty \wedge n < \infty \wedge n \neq 0 \\ 10.0 & \text{otherwise} \end{cases}$$

That is $a_n = \frac{10}{n\pi} \sin n\pi$, which is zero for all values of n.

Determine the expression for the b_n.:

[4]:
```
bn = 2/T*(sp.integrate((50*t + 10)*sp.sin(2*n*sp.pi*t/T), (t, -T/2, 0.0)) +
          sp.integrate(50*t*sp.sin(2*n*sp.pi*t/T), (t, 0.0, T/2)))
display(Markdown("$b_n = " + sp.latex(sp.simplify(bn)) + "$"))
```

$$b_n = \begin{cases} \frac{-10.0 \cos(\pi n) - 10.0 + \frac{20.0 \sin(\pi n)}{\pi n}}{\pi n} & \text{for } n > -\infty \wedge n < \infty \wedge n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We have $b_n = \frac{20}{n^2\pi^2}\sin n\pi - \frac{10}{n\pi}(1+\cos n\pi)$.
Work out the numerical values of the coefficients and then plot the sum of the Fourier series.

[5]:
```
def coeffSawTooth(N, T):
    '''
    Fourier coefficients for saw tooth.
    '''
    a = np.zeros(N)
    b = np.zeros(N)
    a[0] = 5.0
    for n in range(1, N):
        a[n] = 10*np.sin(2*np.pi*n)
        b[n] = 20*np.sin(np.pi*n)/(np.pi**2*n**2) - 10/(np.pi*n)*(1 + np.cos(np.
 ↪pi*n))
    return a, b
#
def fsSum(a, b, N, T, t):
    '''
    Sum of Fourier Series
    '''
    F = a[0]
    for n in range(1, N):
        F = F + a[n]*np.cos(2*n*np.pi*t/T) + b[n]*np.sin(2*n*np.pi*t/T)
    return F
#
N = 10
caSawTooth, cbSawTooth = coeffSawTooth(N, T)
np.set_printoptions(precision = 2)
print(" ")
print("Fourier coefficients for saw tooth.")
print("n\t an\t bn")
for n in range(0, N):
    print("{:d}\t {:.6f}\t {:.6f}".format(n, caSawTooth[n], cbSawTooth[n]))
#
tArrS = np.linspace(tBot, tTop, nT)
fArrS = np.zeros(nT)
for i in range(0, nT):
    fArrS[i] = fsSum(caSawTooth, cbSawTooth, N, T, tArr[i])
print(" ")
plt.plot(tArrS, fArrS, color = 'r', linestyle = '-')
plt.plot(tArr, fArr, color = 'b', linestyle = '--')
plt.title("Wave", size = 14, weight = "bold")
plt.xlabel("time")
plt.ylabel("signal")
plt.axis([tBot, tTop, -0.5, 10.5])
plt.grid(color = 'g')
plt.savefig("SawToothFourier.png")
```

```
Fourier coefficients for saw tooth.
n         an      bn
0         5.000000        0.000000
1         -0.000000       0.000000
2         -0.000000       -3.183099
3         -0.000000       0.000000
4         -0.000000       -1.591549
5         -0.000000       0.000000
6         -0.000000       -1.061033
7         -0.000000       0.000000
8         -0.000000       -0.795775
9         -0.000000       0.000000
```