Fourier Series and Fourier Transforms

- Example from problems class.
- Find the first six terms in the Fourier series for the function with $T=4$ defined by:

  $f(t)=0,$ for $-2\le t<-1,$

  $f(t)=t,$ for $-1\le t<1,$

  $f(t)=0,$ for $1\le t<2.$
- Compute the Fourier transform of the function:

$$g(x)=\begin{cases}1 \text{ if } -\frac{1}{2}\le x<\frac{1}{2},\\ 0 \text{ otherwise.}\end{cases}$$

- Determine the Fourier transform of the function:

$$f(x)=\begin{cases}1+x \text{ if } -1<x\le 0,\\ 1-x \text{ if } 0<x\le 1.\end{cases}$$

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Fourier transform of f(x)

- We have:

$$f(x)=\begin{cases}1+x \text{ if } -1<x\le 0,\\ 1-x \text{ if } 0<x\le 1.\end{cases}$$

$$\tilde{f}(\omega)=\int_{-1}^{0}(1+x)e^{-i\omega x}dx+\int_{0}^{1}(1-x)e^{-i\omega x}dx$$

- Break integral into parts
- First part:

$$A=\int_{-1}^{0}e^{-i\omega x}dx$$
$$=-\frac{1}{i\omega}e^{-i\omega x}\Big|_{-1}^{0}$$
$$=-\frac{1}{i\omega}\left(1-e^{i\omega}\right)$$

- Second part:

$$B=\int_{-1}^{0}xe^{-i\omega x}dx$$
$$=\int_{-1}^{0}xd\left(-\frac{e^{-i\omega x}}{i\omega}\right)$$
$$=-x\frac{e^{-i\omega x}}{i\omega}\Big|_{-1}^{0}+\frac{1}{i\omega}\int_{-1}^{0}e^{-i\omega x}dx$$
$$=-\frac{e^{i\omega}}{i\omega}+\frac{1}{i\omega}A$$
$$=-\frac{e^{i\omega}}{i\omega}-\frac{1}{i\omega}\frac{1}{i\omega}\left(1-e^{i\omega}\right)$$
$$=-\frac{e^{i\omega}}{i\omega}+\frac{1}{\omega^{2}}\left(1-e^{i\omega}\right)$$

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Fourier transform of f(x)

- Third part:

$$C=\int_{0}^{1}e^{-i\omega x}dx$$
$$=-\frac{1}{i\omega}e^{-i\omega x}\Big|_{0}^{1}$$
$$=-\frac{1}{i\omega}\left(e^{-i\omega}-1\right)$$

- Fourth part:

$$D=\int_{0}^{1}xe^{-i\omega x}dx$$
$$=\int_{0}^{1}xd\left(-\frac{e^{-i\omega x}}{i\omega}\right)$$
$$=-x\frac{e^{-i\omega x}}{i\omega}\Big|_{0}^{1}+\frac{1}{i\omega}\int_{0}^{1}e^{-i\omega x}dx$$
$$=-\frac{e^{-i\omega}}{i\omega}-\frac{1}{i\omega}C$$
$$=-\frac{e^{-i\omega}}{i\omega}-\frac{1}{i\omega}\frac{1}{i\omega}\left(e^{-i\omega}-1\right)$$
$$=-\frac{e^{-i\omega}}{i\omega}+\frac{1}{\omega^{2}}\left(e^{-i\omega}-1\right)$$

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Fourier transform of f(x)

- $\tilde{f}(\omega)=A+B+C-D$

$$=-\frac{1}{i\omega}\left(1-e^{i\omega}\right)-\frac{e^{i\omega}}{i\omega}+\frac{1}{\omega^{2}}\left(1-e^{i\omega}\right)-\frac{1}{i\omega}\left(e^{-i\omega}-1\right)+\frac{e^{-i\omega}}{i\omega}-\frac{1}{\omega^{2}}\left(e^{-i\omega}-1\right)$$
$$=-\frac{1}{i\omega}\left(1-e^{i\omega}\right)-\frac{e^{i\omega}}{i\omega}+\frac{1}{\omega^{2}}\left(1-e^{i\omega}\right)-\frac{1}{i\omega}\left(e^{-i\omega}-1\right)+\frac{e^{-i\omega}}{i\omega}-\frac{1}{\omega^{2}}\left(e^{-i\omega}-1\right)$$
$$=-\frac{1}{i\omega}\left(1-e^{i\omega}+e^{i\omega}+e^{-i\omega}-1-e^{-i\omega}\right)+\frac{1}{\omega^{2}}\left(1-e^{i\omega}-e^{-i\omega}+1\right)$$
$$=\frac{2}{\omega^{2}}\left(1-\left(\frac{e^{i\omega}+e^{-i\omega}}{2}\right)\right)$$
$$=\frac{2}{\omega^{2}}\left(1-\cos\omega\right)$$
$$=\frac{4\sin^{2}\frac{\omega}{2}}{\omega^{2}}\ (\text{using } 1-\cos\theta=2\sin^{2}\tfrac{\theta}{2}).$$

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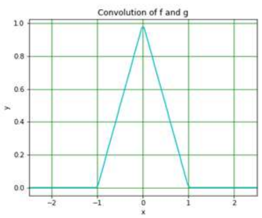
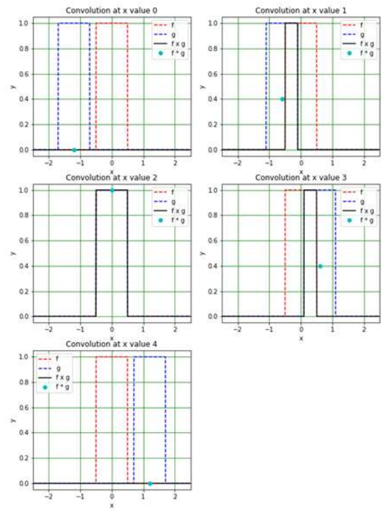
Convolution example

- Remember what convolution is:

  $(f * g)(x) = \int_{-\infty}^{\infty} f(\xi) g(x - \xi) d\xi.$

- The value of $(f * g)(x)$ at a given x is the overlapping area of f and g with $g(\xi) \rightarrow g(-\xi)$.
- (Here we have f = g.)





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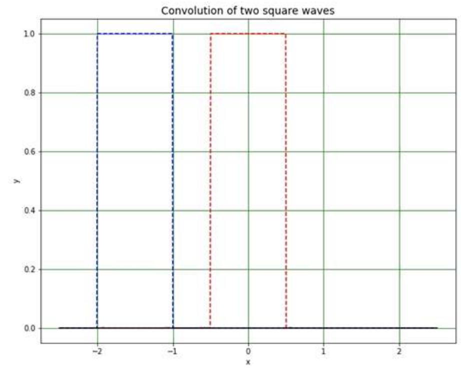
Convolution example – animation



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Convolution example

- If $\mathcal{F}(g)$ is the Fourier Transform of g then:

  $\mathcal{F}(f) = \mathcal{F}(g * g) = \mathcal{F}(g) \times \mathcal{F}(g).$

- This gives us:

$$\tilde{g}(\omega) = \frac{2\sin\frac{\omega}{2}}{\omega}.$$

$$\tilde{f}(\omega) = \frac{4\sin^2\frac{\omega}{2}}{\omega^2}.$$

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