Solving Differential Equations

First order

Separable: $\dfrac{dy}{dx} = f(x)g(y)$

Rewrite and integrate: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$

RHS homogeneous function of degree zero

Use substitution: $y = xu(x),\ \dfrac{dy}{dx} = x\dfrac{du}{dx} + u$

Of form $\dfrac{dy}{dx} + yP(x) = Q(x)$

Use integrating factor: $\text{IF} = \exp\left[\displaystyle\int P(x)\,dx\right]$

Use initial condition to fix arbitrary constant

Solving Differential Equations

Second order homogeneous

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Get roots of auxiliary equation:

$$am^2+bm+c=0$$

m_1, m_2 real and different

$$y=A\exp\left[m_1x\right]+B\exp[m_2x]$$

$m_1=m_2=m$ real

$$y=A\exp\left[mx\right]+Bx\exp[mx]$$

$m_1=\alpha+i\beta,\ m_2=\alpha-i\beta$

$$y=\exp\left[\alpha x\right]\left(P\cos\beta x+Q\sin\beta x\right)$$

Use initial conditions to fix arbitrary constants

Solving Differential Equations

Second order inhomogeneous

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

Solve complementary equation:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$f(x)=px^2+qx+r$$

trial function $y_p=Ax^2+Bx+C$

$$f(x)=A\exp[\gamma x]$$

$\gamma\neq m_1, \gamma\neq m_2$ try $y_p=C\exp[\gamma x]$

$\gamma\neq m_1, \gamma=m_2$ try $y_p=Cx\exp[\gamma x]$

$\gamma=m_1=m_2$ try $y_p=Cx^2\exp[\gamma x]$

$$f(x)=C\cos\gamma x+D\sin\gamma x$$

$\gamma\neq\beta$ try $y_p=A\sin\gamma x+B\cos\gamma x$

$\gamma=\beta$ try $y_p=x\left(A\sin\gamma x+B\cos\gamma x\right)$

Equate coefficients to fix constants in trial function

Add solution of complementary equation to trial function and use initial conditions to fix aribtrary constants

Solving Differential Equations

No explicit y dependence

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0$$

Substitute:

$$v(x) = \frac{dy}{dx}, \quad \frac{d^2y}{dx^2} = \frac{dv}{dx}$$

No explicit x dependence

$$f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0$$

Substitute:

$$v(y) = \frac{dy}{dx}, \quad \frac{d^2y}{dx^2} = v\frac{dv}{dy}$$

Solve as first order equation