Differential equations

- In this lecture we will:
  - Look at second order homogeneous differential equations.
  - Introduce the auxiliary equation and determine its roots.
  - Find out how to solve the homogeneous second order differential equation in the case that the roots of the auxiliary equation are:
    - Real and different.
    - The same.
    - Complex conjugate.
- Some comprehension questions for this lecture.
  - Write down the general form of a homogeneous second order differential equation with constant coefficients.
  - Solve the initial value problem:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0,$$

$$\text{with } y(1) = 1 \text{ and } \frac{dy(1)}{dx} = 1$$

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Homogeneous second order differential equations

- Consider second order homogeneous differential equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

- The coefficients a, b and c are all constants.
- Try to find a solution of the form $y = e^{mx}$.
- Differentiating this gives:

$$\frac{dy}{dx} = me^{mx} \text{ and } \frac{d^2y}{dx^2} = m^2e^{mx}.$$

- Substituting into the original equation we have:

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0,$$

$$\text{or } e^{mx}(am^2 + bm + c) = 0$$

- Now e^{mx} cannot be zero, so:

$$am^2 + bm + c = 0.$$

- This is called the auxiliary equation.
- The above implies that $y = e^{mx}$ is a solution of the differential equation iff (if and only if) m takes one of the values:

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

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Homogeneous second order differential equations

- When the discriminant $b^2 - 4ac > 0$, m_1 and m_2 are real and distinct.
- When $b^2 - 4ac = 0$, the roots are real and equal.
- When $b^2 - 4ac < 0$, the roots are complex conjugate numbers.
- The principle of superposition:
- Supposing we have two solutions of our homogeneous second order differential equation, $y_1(x)$ and $y_2(x)$.
- The sum $C_1y_1(x) + C_2y_2(x)$ is also a solution of the equation.
- Prove this:

$$a\frac{d^2}{dx^2}(C_1y_1 + C_2y_2) + b\frac{d}{dx}(C_1y_1 + C_2y_2) + c(C_1y_1 + C_2y_2)$$

$$= aC_1\frac{d^2y_1}{dx^2} + aC_2\frac{d^2y_2}{dx^2} + bC_1\frac{dy_1}{dx} + bC_2\frac{dy_2}{dx} + cC_1y_1 + cC_2y_2$$

$$= C_1\left(a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right) + C_2\left(a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2\right)$$

$$= 0.$$

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Homogeneous second order differential equations

- Consider various possibilities for the solutions of the auxiliary equation.
- If we have distinct roots, $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$ are linearly independent solutions of our differential equation.
- The functions $y_1(x)$ and $y_2(x)$ are linearly independent if one is not just a multiple of the other, that is: $y_2(x) \neq ky_1(x)$.
- Hence, by the superposition principle, a general solution is: $y(x) = Ae^{m_1x} + Be^{m_2x}$.
- Example:
- Find a general solution of:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0.$$

- The auxiliary equation is:

$$m^2 + 5m - 6 = 0$$

$$\Rightarrow (m+6)(m-1) = 0$$

$$\Rightarrow m_1 = 1 \text{ and } m_2 = -6.$$

- Hence a general solution to the equation is: $y(x) = Ae^{x} + Be^{-6x}$.

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Homogeneous second order differential equations

- Another example, solve the initial value problem

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-y=0$$
$$y(0)=0$$
$$\frac{dy(0)}{dx}=-1 \text{ or } y'(0)=-1.$$

- The auxiliary equation is:

$$m^2+2m-1=0$$
$$\Rightarrow m_1=\frac{-2+\sqrt{8}}{2}=-1+\sqrt{2}$$
$$\text{and } m_2=\frac{-2-\sqrt{8}}{2}=-1-\sqrt{2}.$$

- A general solution is

$$y(x)=Ae^{(-1+\sqrt{2})x}+Be^{(-1-\sqrt{2})x}.$$

- The initial conditions can be used to determine A and B:

$$y(0)=Ae^0+Be^0$$
$$\Rightarrow 0=A+B \text{ or } A=-B$$
$$\frac{dy}{dx}=A\left(-1+\sqrt{2}\right)e^{(-1+\sqrt{2})x}+B\left(-1-\sqrt{2}\right)e^{(-1-\sqrt{2})x}$$
$$\frac{dy(0)}{dx}=\left(-1+\sqrt{2}\right)A+\left(-1-\sqrt{2}\right)B$$
$$-1=\left(-1+\sqrt{2}\right)A-\left(-1-\sqrt{2}\right)A$$
$$\Rightarrow -1=2\sqrt{2}A \text{ or } A=-\frac{1}{2\sqrt{2}}.$$

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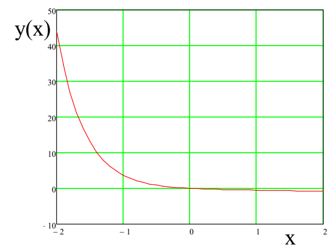
Homogeneous second order differential equations

- Rewriting:

$$A=-\frac{\sqrt{2}}{4} \text{ and } B=\frac{\sqrt{2}}{4}.$$

- Putting this together:

$$y(x)=-\frac{\sqrt{2}}{4}e^{(-1+\sqrt{2})x}+\frac{\sqrt{2}}{4}e^{(-1-\sqrt{2})x}.$$



- If the roots of auxiliary equation are the same (m), we can use $y = e^{mx}$ and $y = xe^{mx}$ as two linearly independent solutions of the differential equation.
- Example:
- Find a general solution of:

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0.$$

- Auxiliary equation:

$$m^2+4m+4=0$$
$$\Rightarrow (m+2)^2=0$$
$$m_1=m_2=-2.$$

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Homogeneous second order differential equations

- General solution therefore:

$$y(x)=Ae^{-2x}+Bxe^{-2x}$$

- What if the auxiliary equation has complex conjugate roots $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$?
- Then:

$$y=Ae^{(\alpha+i\beta)x}+Be^{(\alpha-i\beta)x}$$
$$=e^{\alpha x}\left(Ae^{i\beta x}+Be^{-i\beta x}\right)$$
$$=e^{\alpha x}\begin{pmatrix}A\left(\cos(\beta x)+i\sin(\beta x)\right)\\+B\left(\cos(-\beta x)+i\sin(-\beta x)\right)\end{pmatrix}$$
$$=e^{\alpha x}\begin{pmatrix}(A+B)\cos(\beta x)\\+i(A-B)\sin(\beta x)\end{pmatrix}.$$

- Writing $P = A + B$ and $Q = i(A - B)$, we then have the general solution:

$$y(x)=e^{\alpha x}\left(P\cos(\beta x)+Q\sin(\beta x)\right).$$

- Example:
- Find the general solution of:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=0.$$

- Auxiliary equation:

$$m^2+2m+4=0$$
$$m_1=\frac{-2+\sqrt{4-16}}{2},\ m_2=\frac{-2-\sqrt{4-16}}{2}$$
$$\Rightarrow m_1=-1+\sqrt{-3},\ m_2=-1-\sqrt{-3}$$
$$\text{or } m_1=-1+i\sqrt{3} \text{ and } m_2=-1-i\sqrt{3}.$$

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Homogeneous second order differential equations

- So, with $\alpha=-1$ and $\beta=\sqrt{3}$:

$$y=Pe^{-x}\cos\sqrt{3}x+Qe^{-x}\sin\sqrt{3}x.$$

- Another example:
- Solve the initial value problem:

$$\frac{d^2y}{dx^2}+4y=0,\ y(0)=1,\ \frac{dy(0)}{dx}=0.$$

- Auxiliary equation:

$$m^2+4=0$$
$$\Rightarrow m_1=2i,\ m_2=-2i.$$

- Hence $\alpha=0,\ \beta=2.$
- This gives:

$$y=P\cos 2x+Q\sin 2x,$$
$$\frac{dy}{dx}=-2P\sin 2x+2Q\cos 2x.$$

- Using the initial conditions we have:

$$y(0)=P=1,$$
$$\frac{dy(0)}{dx}=2Q=0, \text{ so } Q=0.$$

- The required solution is therefore:

$$y(x)=\cos 2x.$$

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