Differential equations

- In this lecture we will:
  - Introduce differential equations.
  - Look at how differential equations can be classified.
  - Learn how to find solutions to ordinary first order differential equations.
- Some comprehension questions for this lecture.
  - What is the order of the following equation:
    $$7\frac{d^2y}{dx^2}+2y+3=0$$
  - Is it linear? Is it homogeneous?
  - The number of radioactive decays per unit time in a sample is proportional to the number, N, of nuclei that could potentially decay, i.e. it obeys the equation:
    $$\frac{dN}{dt}=-\lambda N.$$
    Solve this equation.

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Differential equations

- We will work in 1D in this section!
- "Normal" equations relate an independent variable (often labelled x) to a dependent variable (often y).
- E.g. $y = \cos x - {}^{1}/_{3}$.
- Solutions of these equations (e.g. for y = 0) are typically a number, or a collection of numbers.
- Ordinary differential equations (ODEs) relate an independent variable (x) to a dependent variable (y) and one or more of its derivatives with respect to the independent variable.
- Example $\frac{d^3y}{dx^3}+\sin x\frac{dy}{dx}=5x^2$.
- The solution of a differential equation is typically one or more functions which relate the dependent variable to the independent variable.
- DEs are the natural way of representing many physical laws and effects, e.g. Newton's second law, Maxwell's equations (partial derivatives, 3D!), radioactivity, etc.
- The <u>order</u> of a differential equation is the highest derivative that appears in the equation.
- E.g. $\frac{dy}{dx}-y=0$ is 1^{st} order.
  $x^2\frac{d^5y}{dx^5}-\sin x\frac{dy}{dx}=0$ is 5^{th} order.

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Differential equations

- ODEs are <u>linear</u> if the dependent variable (y) and its derivatives appear to the power 1.
- E.g. $\frac{dy}{dx}=ky+x^3$ is linear.
  $L\frac{d^2y}{dx^2}+g\sin y=0$ is non-linear.
- ODEs are <u>homogeneous</u> if each term contains either the dependent variable (y) or one of its derivatives.
- E.g. $\frac{dy}{dx}=Ay$ is homogeneous.
  $A\frac{dy}{dx}+By+C=0$ is inhomogeneous.
- Classify these equations:
- $\frac{dy}{dx}=ky+x^2$
- $\frac{d^2u}{dt^2}+\omega^2t=0$
- $\frac{dx}{dt}=x^2+1$
- $\frac{d^2u}{dx^2}-x\frac{du}{dx}+u=0$
- $\frac{d^2u}{dx^2}-x\frac{du}{dx}+u=17u$
- $\frac{d^2u}{dx^2}-x\left(\frac{du}{dx}\right)^2+u=17u-32$

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Solving first order differential equations

- Most obvious is the solution to equations of the form:
  $$\frac{dy}{dx}=f(x)$$
- All we need to do is take the "anti-derivative" (integral) of both sides:
  $$\int\frac{dy}{dx}dx=\int f(x)\,dx$$
  $$\Rightarrow y=\int f(x)\,dx$$
- Note, this DE may also be written...
  $$dy=f(x)\,dx$$
- ...and its solution:
  $$\int dy=\int f(x)\,dx$$
  $$\Rightarrow y=\int f(x)\,dx$$
- This works as long as we can integrate f(x).
- There are many functions for which there is no exact and closed form for the integral!
- Also easy to solve are equations like:
  $$\frac{dy}{dx}=g(y)\text{ or }dy=g(y)\,dx$$
  $$\Rightarrow\int dx=\int\frac{1}{g(y)}dy$$

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Solving first order DEs

- An example:
$$\frac{dy}{dx} = 6x^2 - 2x + 5$$
$$y = \int 6x^2 - 2x + 5\, dx$$
$$= 2x^3 - x^2 + 5x + c$$
- Note constant of integration, c!
- Another example:
$$\frac{dy}{dx} = y \Rightarrow dy = y\, dx$$
$$\text{and } \int \frac{dy}{y} = \int dx$$
- Integrate and add constant:
$\ln y = x + c$ or $y = \exp(x + c)$.
Hence $y = \exp(x)\exp(c) = A\exp(x)$

Solving first order DEs – separating variables

- Equations of the form...
$$\frac{dy}{dx} = f(x)g(y)$$
- ...can be solved by separating the variables:
$$\frac{dy}{g(y)} = f(x)\, dx$$
$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx$$
- Equation might need simplification to show that it is separable.

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Solving first order DEs – separating variables

- An example:
$$\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2 y - x^2}$$
$$= \frac{y^2(1+x)}{x^2(y-1)}$$
$$= \frac{1+x}{x^2}\frac{y^2}{(y-1)}$$
- Hence:
$$\int \frac{y-1}{y^2}\, dy = \int \frac{1+x}{x^2}\, dx$$
$$\int \frac{1}{y} - y^{-2}\, dy = \int x^{-2} + \frac{1}{x}\, dx$$
$$\ln y + \frac{1}{y} = -\frac{1}{x} + \ln x + c$$

Solving first order DEs – substitution

- All equations of the form...
$$\frac{dy}{dx} = f(x, y)$$
- ...in which $f(x, y)$ is a homogeneous function of degree zero, that is $f(tx, ty) = f(x, y)$, can be solved by first substituting $y(x) = u(x) \times x$.
- We then have (differentiate product):
$$\frac{dy}{dx} = \frac{d(ux)}{dx} = x\frac{du}{dx} + u$$
- Substitute in the above equation:
$$x\frac{du}{dx} + u = f(x, ux)$$

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Solving first order DEs – substitution

- Now set $t = 1/x$:
$$x\frac{du}{dx} + u = f\left(\frac{1}{x}x, \frac{1}{x}ux\right) = f(1, u)$$
- Can separate the variables to solve:
$$x\frac{du}{dx} + u = f(1, u)$$
$$\frac{du}{dx} = \frac{f(1,u) - u}{x}$$
$$\int \frac{du}{f(1,u) - u} = \int \frac{dx}{x}.$$
- An example: $\frac{dy}{dx} = \frac{x + 3y}{2x}$
- Substitute $y = ux$:
$$x\frac{du}{dx} + u = \frac{x + 3ux}{2x}$$
$$\Rightarrow x\frac{du}{dx} = \frac{x + 3ux}{2x} - u = \frac{1}{2} + \frac{u}{2}$$
$$\Rightarrow \int \frac{2}{1+u}\, du = \int \frac{dx}{x}$$
$$\Rightarrow 2\ln(1+u) = \ln x + c.$$
$$\text{Using } u = \frac{y}{x}$$
$$2\ln\left(1 + \frac{y}{x}\right) = \ln x + c$$

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Solving first order DEs – integrating factor

- A further method is using an integrating factor.
- Applies to equations of the form
$$\frac{dy}{dx} + yP(x) = Q(x)$$
- The integrating factor (IF) is $\exp\left(\int P(x)\, dx\right)$.
- Multiply equation by IF:
$$e^{\int P(x)dx}\left(\frac{dy}{dx} + yP(x)\right) = e^{\int P(x)dx} Q(x).$$
- Expanding:
$$e^{\int P dx}\frac{dy}{dx} + e^{\int P dx} yP(x) = e^{\int P dx} Q(x)$$
- Hence (integrating w.r.t. x):
$$\int e^{\int P dx}\, dy + \int e^{\int P dx} yP\, dx = \int e^{\int P dx} Q\, dx$$
- Integrate first term by parts, i.e. use $\int u\, dv = uv - \int v\, du$.
$$\int e^{\int P dx}\, dy = e^{\int P dx} y - \int y\, d\left(e^{\int P dx}\right)$$
$$= y e^{\int P dx} - \int yP e^{\int P dx}\, dx$$
- Inserting back into top equation:
$$y e^{\int P dx} - \int yP e^{\int P dx}\, dx + \int e^{\int P dx} yP\, dx = \int e^{\int P dx} Q\, dx$$

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Solving first order DEs – integrating factor

- Hence we have:

$$y e^{\int P\,dx} = \int e^{\int P\,dx} Q\,dx$$

$$\Rightarrow y = e^{-\int P\,dx} \int e^{\int P\,dx} Q\,dx$$

- An example:

$$\frac{dy}{dx} + 5y = e^{2x}$$

- IF is $e^{\int 5\,dx} = e^{5x}$.
- Multiply through by IF:

$$e^{5x}\frac{dy}{dx} + 5ye^{5x} = e^{2x}e^{5x}$$

- The LHS is:

$$\frac{d}{dx} ye^{5x} = 5ye^{5x} + e^{5x}\frac{dy}{dx}$$

- So we see that we can solve this equation using

$$\frac{d}{dx} y e^{5x} = e^{2x}e^{5x}$$

$$ye^{5x} = \int e^{7x}dx$$

$$= \frac{e^{7x}}{7} + c$$

- Hence:

$$y = \frac{e^{7x-5x}}{7} + ce^{-5x} = \frac{e^{2x}}{7} + ce^{-5x}$$

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Integrating factors – alternative derivation

- If have equation

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x)$$

such that

$$\frac{d}{dx}a_1 y = a_1\frac{dy}{dx} + y\frac{da_1}{dx}$$

$$= a_1\frac{dy}{dx} + a_0 y$$

- That is, if $\frac{da_1}{dx} = a_0$.
- Then can rewrite and easily solve:

$$\frac{d}{dx}a_1(x)\,y = b(x) \Rightarrow a_1(x)y = \int b(x)\,dx$$

and $y = \frac{1}{a_1(x)}\int b(x)\,dx.$

- To solve more general equations

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

- Look for an integrating factor I(x) which equation can be multiplied by...

$$I(x)\frac{dy}{dx} + I(x)P(x)\,y = I(x)Q(x)$$

- Choose this so that:

$$\frac{d}{dx}I(x) = I(x)P(x)\,y$$

- To see why, cf. condition

$$\frac{d}{dx}a_1(x) = a_0(x)$$

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Integrating factors – alternative derivation

- But this is a separable equation:

$$\frac{dI(x)}{I(x)} = P(x)\,dx$$

$$\int\frac{dI(x)}{I(x)} = \int P(x)\,dx$$

$$\ln\left(I(x)\right) = \int P(x)\,dx$$

$$I(x) = \exp\left(\int P(x)\,dx\right)$$

- Our DE now becomes

$$\frac{d}{dx}I(x)\,y = I(x)Q(x)$$

- Which can be integrated to give:

$$I(x)\,y = \int I(x)Q(x)\,dx$$

$$\Rightarrow y = \frac{1}{I(x)}\int I(x)Q(x)\,dx$$

$$\text{or } y = e^{-\int P(x)dx}\int e^{\int P(x)dx}Q(x)\,dx$$

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