

Differential equations	
 ODEs are <u>linear</u> if the dependent variable (y) and its derivatives appear to the power 1. E.g. dy/dx = ky + x³ is linear. L d²y/dx² + g sin y = 0 is non-linear. ODEs are <u>homogeneous</u> if each term contains either the dependent variable (y) or one of its derivatives. E.g. dy/dx = Ay is homogeneous. A dy/dx + By + C = 0 is inhomogeneous. 	Classify these equations: $\frac{dy}{dx} = ky + x^{2}$ $\frac{d^{2}u}{dt^{2}} + \omega^{2}t = 0$ $\frac{dx}{dt} = x^{2} + 1$ $\frac{d^{2}u}{dx^{2}} - x\frac{du}{dx} + u = 0$ $\frac{d^{2}u}{dx^{2}} - x\frac{du}{dx} + u = 17u$ $\frac{d^{2}u}{dx^{2}} - x\left(\frac{du}{dx}\right)^{2} + u = 17u - 32$

Solving first order differential equations

Most obvious is the solution to equations of the form:

 $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$

 $\int \frac{dy}{dx} dx = \int f(x) dx$ $\Rightarrow y = \int f(x) dx$

dy = f(x) dx $dy = \int f(x) dx$ $dy = \int f(x) dx$ $dy = \int f(x) dx$

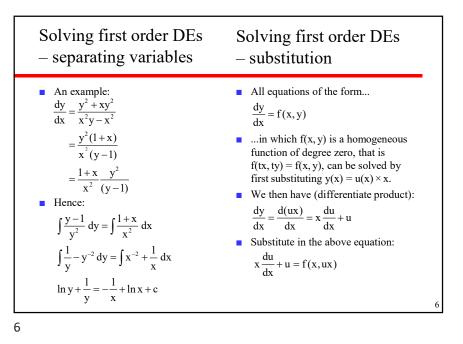
 All we need to do is take the "antiderivative" (integral) of both sides:

Note, this DE may also be written...

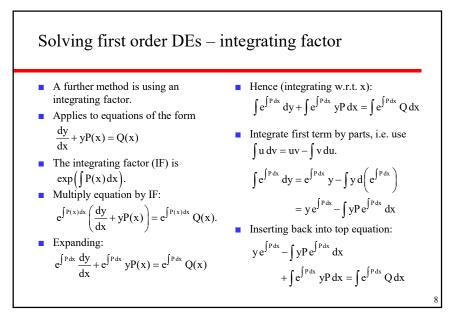
- This works as long as we can integrate f(x).
- There are many functions for which there is no exact and closed form for the integral!
- Also easy to solve are equations like:

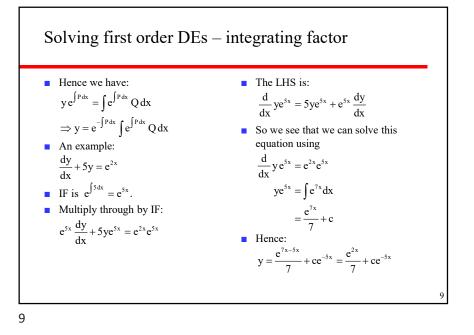
$$\frac{dy}{dx} = g(y) \text{ or } dy = g(y)dx$$
$$\Rightarrow \int dx = \int \frac{1}{g(y)}dy$$

Solving first order DEs	Solving first order DEs – separating variables
An example: $\frac{dy}{dx} = 6x^2 - 2x + 5$ $y = \int 6x^2 - 2x + 5 dx$ $= 2x^3 - x^2 + 5x + c$ Note constant of integration, c! Another example: $\frac{dy}{dx} = y \Rightarrow dy = y dx$ and $\int \frac{dy}{y} = \int dx$ Integrate and add constant: $\ln y = x + c \text{ or } y = \exp(x + c).$ Hence $y = \exp(x)\exp(c) = A \exp(x)$	 Equations of the form ^{dy}/_{dx} = f(x)g(y) can be solved by separating the variables: ^{dy}/_{g(y)} = f(x)dx ⇒ ∫ ^{dy}/_{g(y)} = ∫ f(x)dx Equation might need simplification to show that it is separable.



Solving first order DEs – substitution		
Now set $t = 1/x$: $x \frac{du}{dx} + u = f\left(\frac{1}{x}x, \frac{1}{x}ux\right) = f(1, u)$ Can separate the variables to solve: $x \frac{du}{dx} + u = f(1, u)$ $\frac{du}{dx} = \frac{f(1, u) - u}{x}$ $\int \frac{du}{f(1, u) - u} = \int \frac{dx}{x}.$	• An example: $\frac{dy}{dx} = \frac{x+3y}{2x}$ • Substitute y = ux: $x \frac{du}{dx} + u = \frac{x+3ux}{2x}$ $\Rightarrow x \frac{du}{dx} = \frac{x+3ux}{2x} - u = \frac{1}{2} + \frac{u}{2}$ $\Rightarrow \int \frac{2}{1+u} du = \int \frac{dx}{x}$ $\Rightarrow 2 \ln(1+u) = \ln x + c.$ Using $u = \frac{y}{x}$ $2 \ln \left(1 + \frac{y}{x}\right) = \ln x + c$ 7	





Integrating factors - alternative derivation To solve more general equations If have equation $a_1(x)\frac{dy}{dx}^{1} + a_0(x)y = b(x)$ $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x) \, y = Q(x).$ such that ■ Look for an integrating factor I(x) $\frac{d}{dx}a_1y = a_1\frac{dy}{dx} + y\frac{da_1}{dx}$ which equation can be multiplied by... $=a_1\frac{dy}{dx}+a_0y$ $I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$ • That is, if $\frac{da_1}{dx} = a_0$. • Choose this so that: $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{I}(x) = \mathrm{I}(x)\mathrm{P}(x)\,\mathrm{y}$ Then can rewrite and easily solve: To see why, cf. condition $\frac{d}{dx}a_1(x)y = b(x) \Longrightarrow a_1(x)y = \int b(x)dx$ $\frac{d}{dx}a_1(x) = a_0(x)$ and $y = \frac{1}{a_1(x)} \int b(x) dx$. 10

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