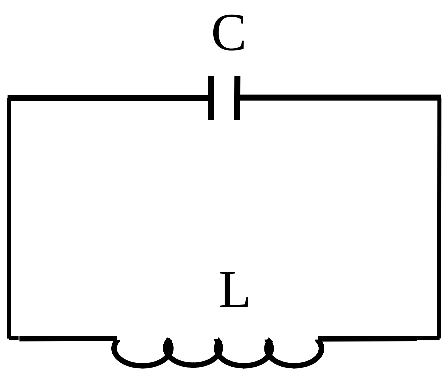
Lecture 18

- In this lecture we will look at:
  - Circuits containing capacitance and inductance.
  - Charge current and energy oscillations.
  - LCR circuits.
  - Damped charge, current and energy oscillations.

- After this lecture, you should be able to answer the following questions:
- Write down the differential equation describing the charge oscillations of a circuit containing an inductance and a capacitance.
- How does this equation change if a series resistance is introduced to the circuit?
- Explain the behaviour of a lightly damped series LCR circuit, starting from the point at which the capacitor is charged and there is no current flowing in the circuit.

Electromagnetic Oscillations

- Look at circuits containing both capacitance and inductance.

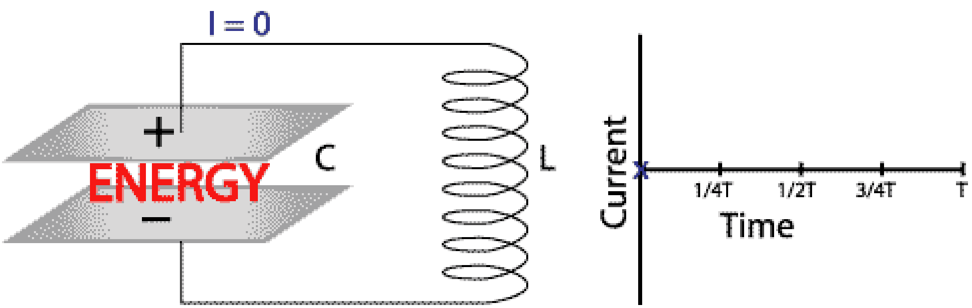


- Have shown that energy stored in a capacitor and inductor given by:
  - $U_E = \frac{q^2}{2C}.$
  - $U_B = \frac{Li^2}{2}.$

- If start with capacitor charged, have energy in E field, energy density: $u_E = \frac{1}{2}\varepsilon_0 E^2.$
- Capacitor will discharge through inductor.
- Current will cause build up of magnetic field, energy then stored in B field, energy density:
  $u_B = \frac{B^2}{2\mu_0}.$
- B field will then decay, inducing current and charging up capacitor...
- Note, using small letters for varying quantities (e.g. charge q) and large letters for constants (e.g. amplitude of charge oscillations, Q).

Electromagnetic Oscillations

- 1



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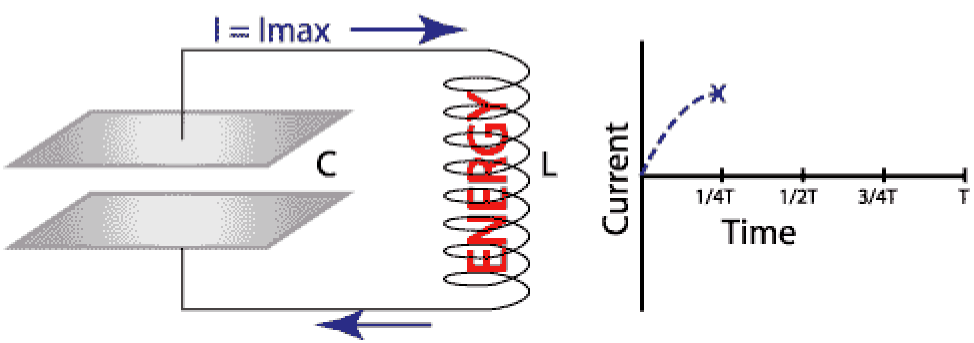

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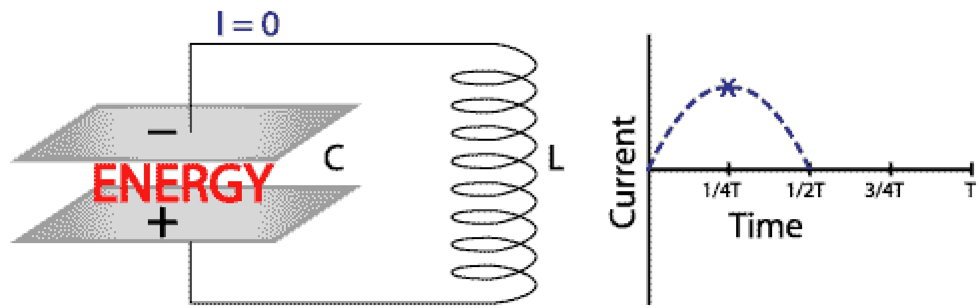

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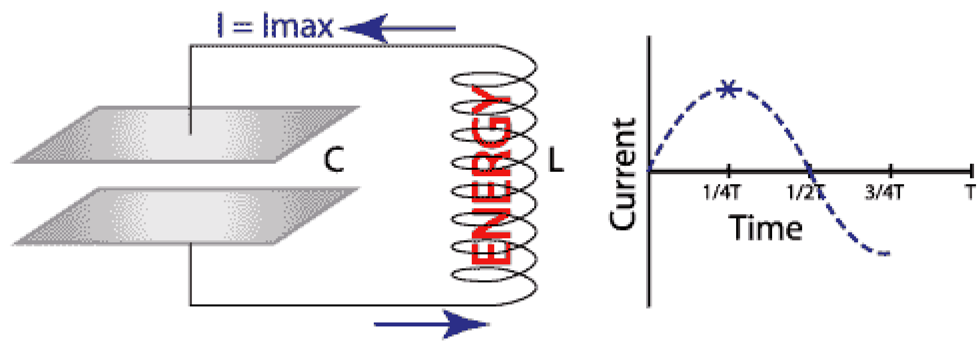


Energy in LC Circuit

- Total energy in L and C is:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}.$$

- As no resistance in circuit, no energy is dissipated, that is:

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0.$$

- Making the substitutions:

$$i = \frac{dq}{dt} \text{ and } \frac{di}{dt} = \frac{d^2q}{dt^2},$$

we then have:

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad [18.1]$$

- This differential equation describes the oscillations of the LC circuit.

- Try solution $q = Q\cos(\omega t + \phi)$ [18.2]
- Calculate derivatives:

$$\frac{dq}{dt} = -Q\omega\sin(\omega t + \phi),$$

$$\frac{d^2q}{dt^2} = -Q\omega^2\cos(\omega t + \phi).$$

- Substitute into differential equation:

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

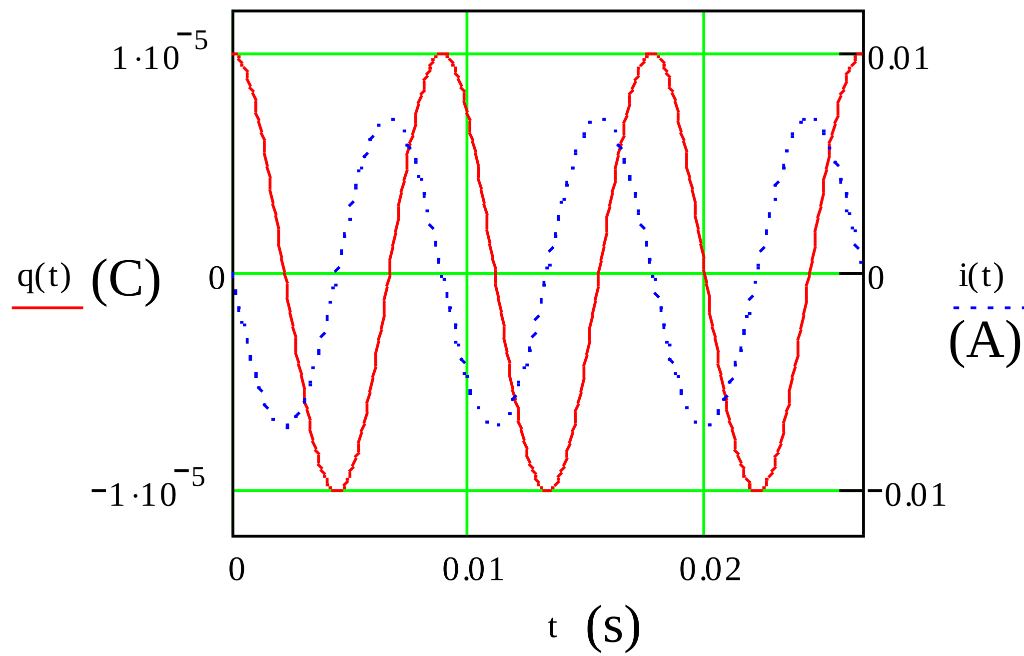
$$\Rightarrow -\omega^2 LQ\cos(\omega t + \phi) + \frac{Q}{C}\cos(\omega t + \phi) = 0$$

$$\text{or } \left(-\omega^2 L + \frac{1}{C}\right)Q\cos(\omega t + \phi) = 0$$

$$\Rightarrow \omega^2 = \frac{1}{LC} \qquad [18.3]$$

Charge and Current Oscillations

- Hence $q = Q\cos(\omega t + \phi)$ is solution of $L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$ provided $\omega = \frac{1}{\sqrt{LC}}$.
- Thus the charge stored on the capacitor exhibits sinusoidal oscillations with frequency $f = \frac{1}{2\pi\sqrt{LC}}$.
- Current in circuit is given by
$$i = \frac{dq}{dt} = -Q\omega\sin(\omega t + \phi) \qquad [18.4]$$
- Hence current also shows sinusoidal oscillations, with amplitude $Q\omega$.
- Graph of charge stored on capacitor and current in circuit, with $C = 10^{-6}$ F, L = 2 H (gives f = 112 Hz), charge on capacitor at t = 0 is 10^{-5} C (i.e. initial potential 10 V):



Energy Oscillations

- Can now check that total energy is conserved.
- Look at energy stored in E field in capacitor...

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi)$$

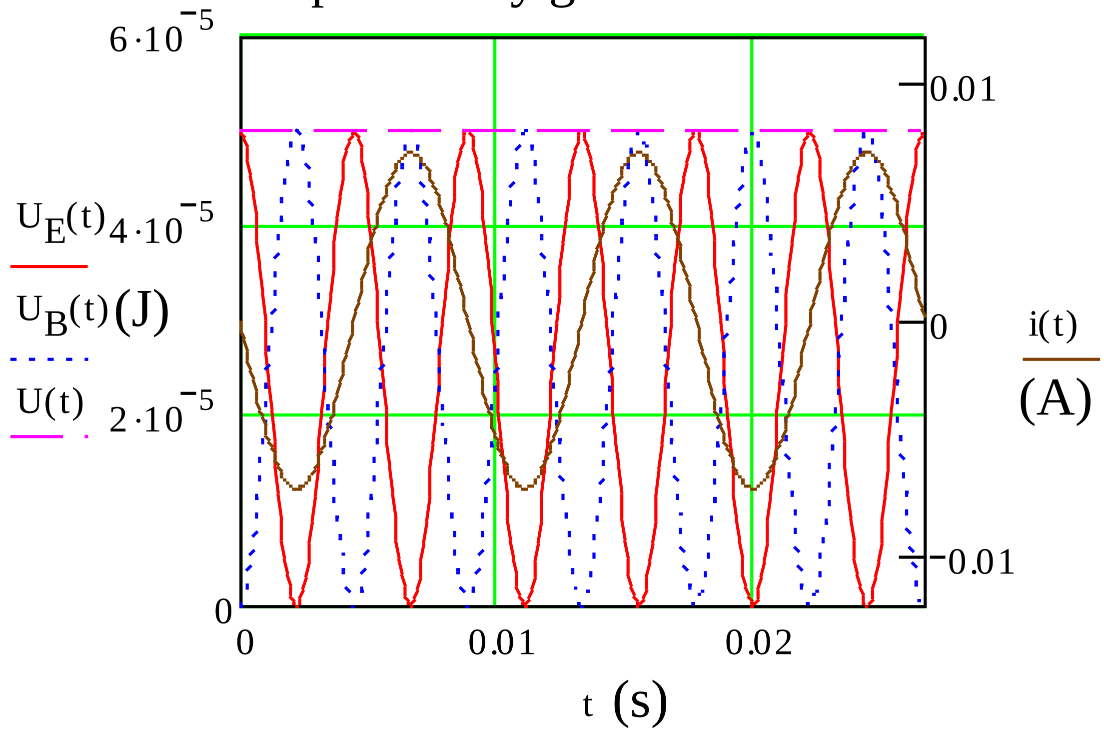
- ...and at energy stored in B field in inductor:

$$U_B = \frac{Li^2}{2} = \frac{L\omega^2 Q^2 \sin^2(\omega t + \phi)}{2}.$$

- Substituting for ω gives:

$$U_B = \frac{Li^2}{2} = \frac{Q^2}{2C}\sin^2(\omega t + \phi).$$

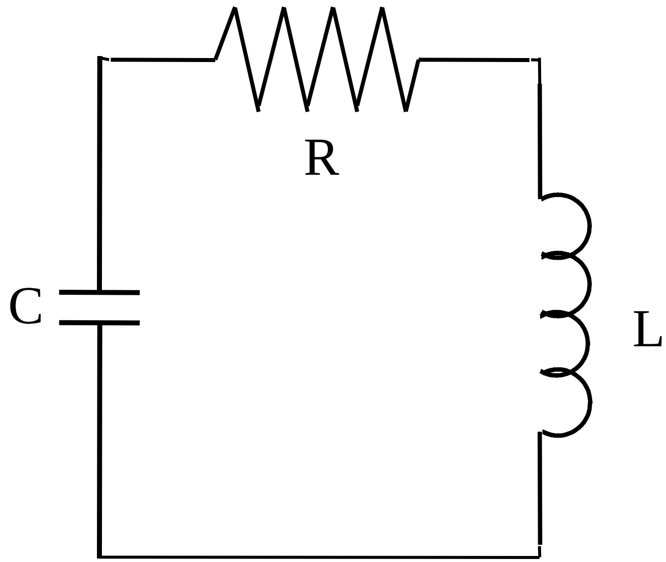
- Plotting these quantities and their sum, U, for the values of C, L and Q used previously gives:



- Notice total energy looks constant as expected: exercise, prove this!

RLC Circuit

- Now include resistance in the circuit:



- The total electromagnetic energy, $U = U_E + U_B$, is now no longer constant as it is converted to heat in the resistor at a rate given by i^2R.

- That is:

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

- Substituting for the current as before:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \qquad [18.5]$$

- The solution to this equation is:

$$q = Q\exp\left[-\frac{Rt}{2L}\right]\cos(\omega' t + \phi),$$

where $\omega' = \sqrt{\omega^2 - (R/2L)^2}$ [18.6]

- The amplitude of the charge oscillations decreases with time (i.e. they are damped) according to $Q\exp\left[-\frac{Rt}{2L}\right]$.

Damped Charge and Current Oscillations

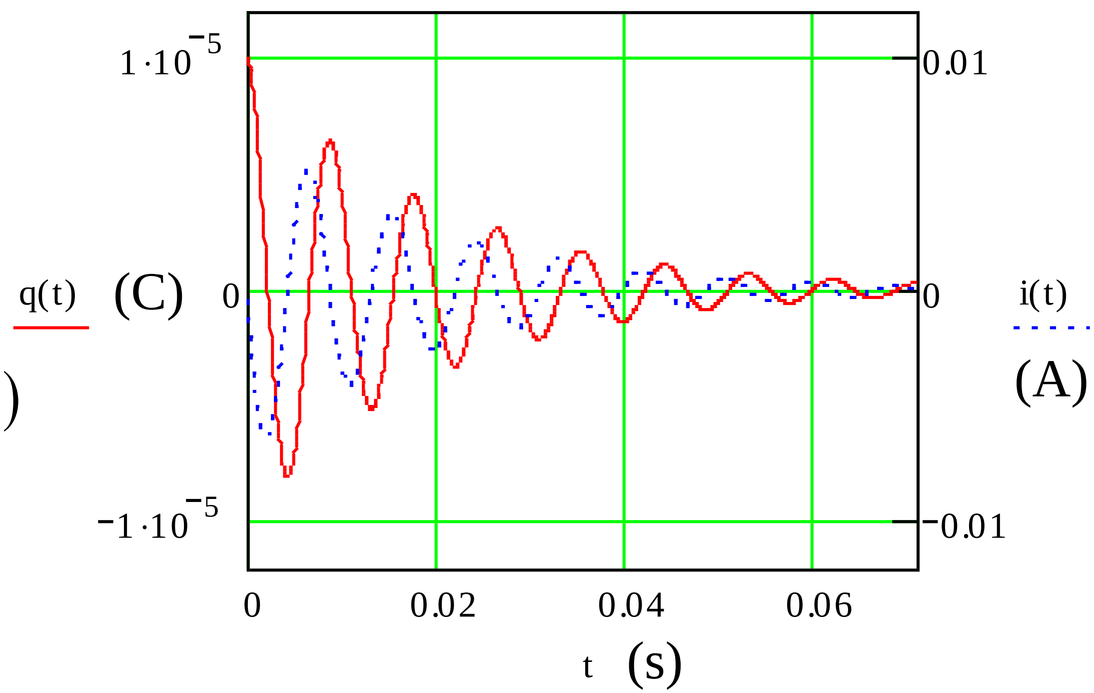
- The (angular) frequency of the oscillations is also decreased.
- Consider situations in which $\omega \approx \omega'$, this is the case when the damping is not too strong.
- The current is given by:

$$\frac{dq}{dt} = \frac{d}{dt} Q \exp\left[-\frac{Rt}{2L}\right] \cos(\omega' t + \phi)$$

$$= -\frac{R}{2L} Q \exp\left[-\frac{Rt}{2L}\right] \cos(\omega' t + \phi)$$

$$-\omega' Q \exp\left[-\frac{Rt}{2L}\right] \sin(\omega' t + \phi).$$

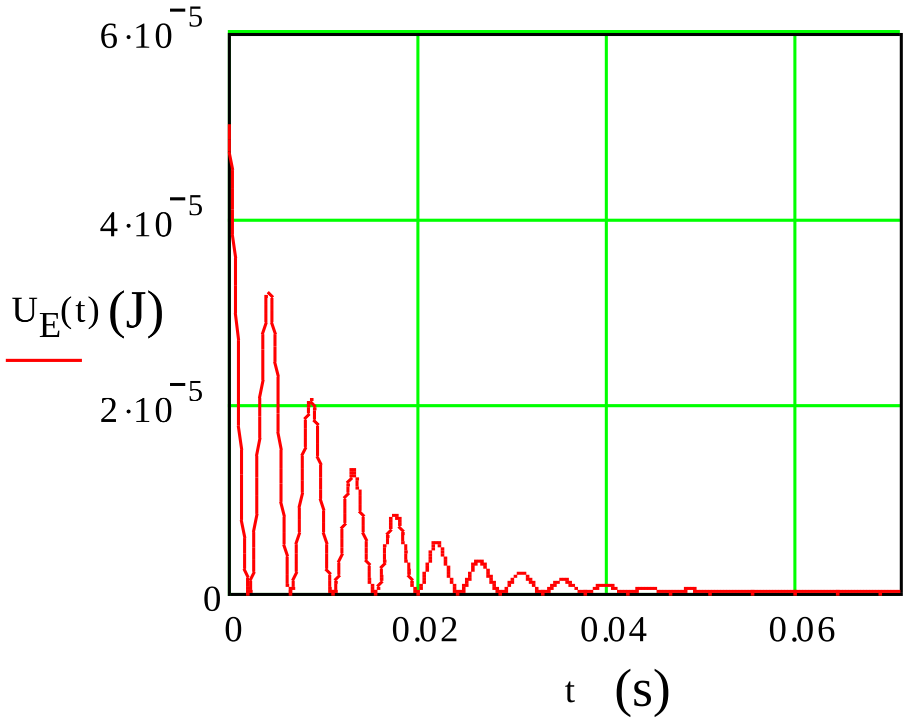
- With the same L and C values as before, adding a 200 Ω resistor, the current and charge behave as below:



Damped Energy Oscillations

- The energy stored in the electric field of the capacitor is given by:

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\exp\left[-\frac{Rt}{L}\right]\cos^2(\omega' t + \phi).$$



- The expression for U_B is a little messy, but adding this and the total electromagnetic energy U to the plot gives: