Lecture 4

- This lecture, we will look at:
  - Gauss' Law and its equivalence to Coulomb's Law
  - Some applications of Gauss' Law
  - Gauss' Law and the Shell Theorem

- After this lecture, you should be able to answer the following questions:
- What is the charge enclosed by a surface through which the electric flux is 24 Nm^2C^{-1}?
- Is this charge positive or negative?
- Calculate the electric field due to an infinite line of charge with linear density corresponding to one electron every nanometre.
- Use Gauss Law to prove the Shell Theorem.

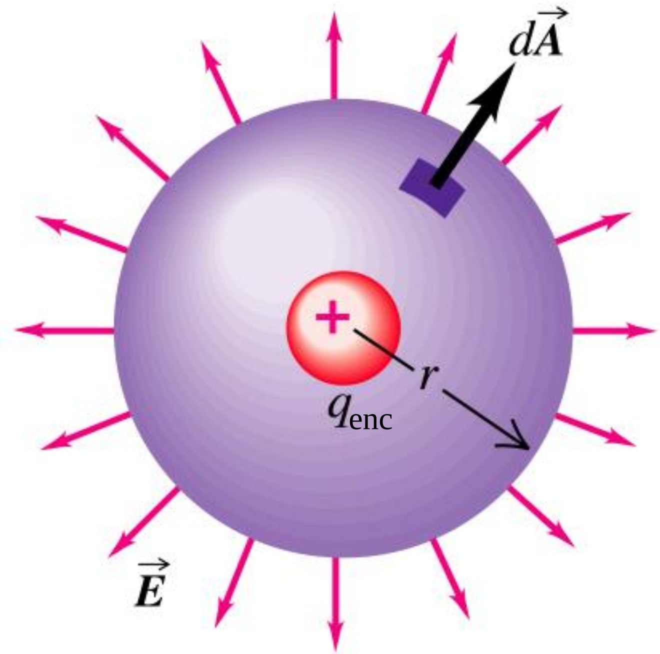
Gauss' Law: Equivalence to Coulomb's Law

- Gauss' Law (in a vacuum) states: The flux of the electric field out of any closed surface equals the charge enclosed divided by ε_0:

$$\Phi = \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} \qquad [4.1]$$

- We now show that Gauss' Law is equivalent to Coulomb's Law.
- Consider point charge q_{enc} at centre of spherical Gaussian surface.
- Angle between $d\vec{A}$ and $\vec{E}$ is always zero, hence:

$$\Phi = \oint_S \vec{E} \cdot d\vec{A} = \oint_S E\, dA = \frac{q_{enc}}{\varepsilon_0}$$



- E is constant over surface, so have:

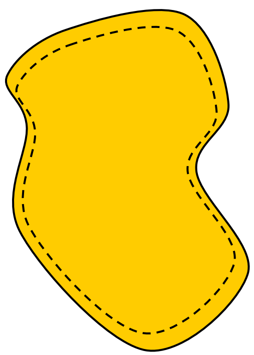
$$\oint_S E\, dA = E \oint_S dA = E(4\pi r^2).$$

- Hence $E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0}$ or $E = \frac{1}{4\pi\varepsilon_0}\frac{q_{enc}}{r^2}$,

as required by Coulomb's Law.

Field Inside Charged Isolated Conductor

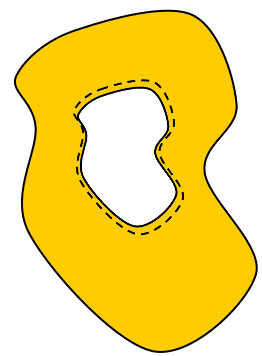
- The E field in a charged isolated conductor must be zero. (Why?)
- Use Gauss' Law to find out where the charge resides.
- Consider lump of copper, draw Gaussian surface just inside surface of copper.



- Gauss' Law says that for any closed surface:

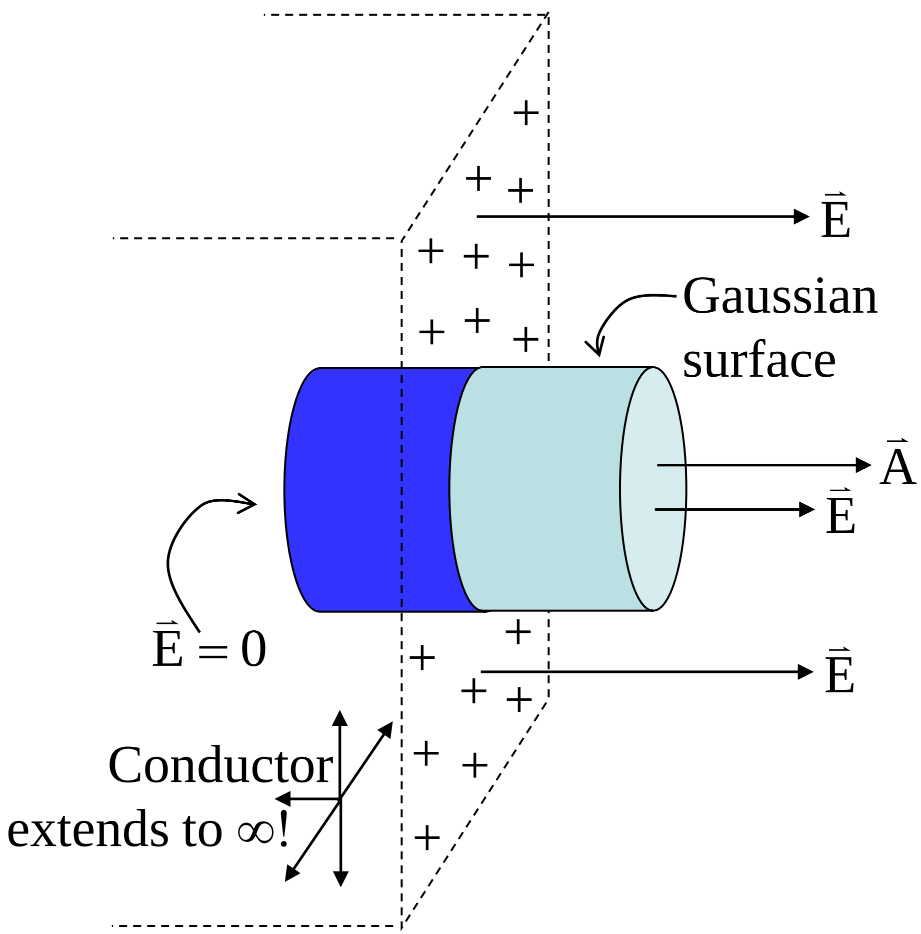
$$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}.$$

- As E field zero, $q_{enc}/\varepsilon_0 = 0$, hence no charge inside the Gaussian surface: it must all be on the surface of the copper.
- Repeat argument after having scooped out a cavity in the copper.
- No charge inside Gaussian surface, so no charge on cavity walls.



Field Outside Infinite Charged Conductor

- Surface charge density σ ($\mathrm{C\,m^{-2}}$).



- Using Gauss' Law:

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}.$$

- For curved surfaces:

$$\vec{E} \text{ and } d\vec{A} \text{ at } 90^\circ \Rightarrow \vec{E}\cdot d\vec{A} = 0.$$

- For RH end:

$$\vec{E} \text{ and } d\vec{A} \text{ parallel} \Rightarrow \vec{E}\cdot d\vec{A} = E\,dA.$$

$$E \text{ const, so } \int_S E\,dA = EA.$$

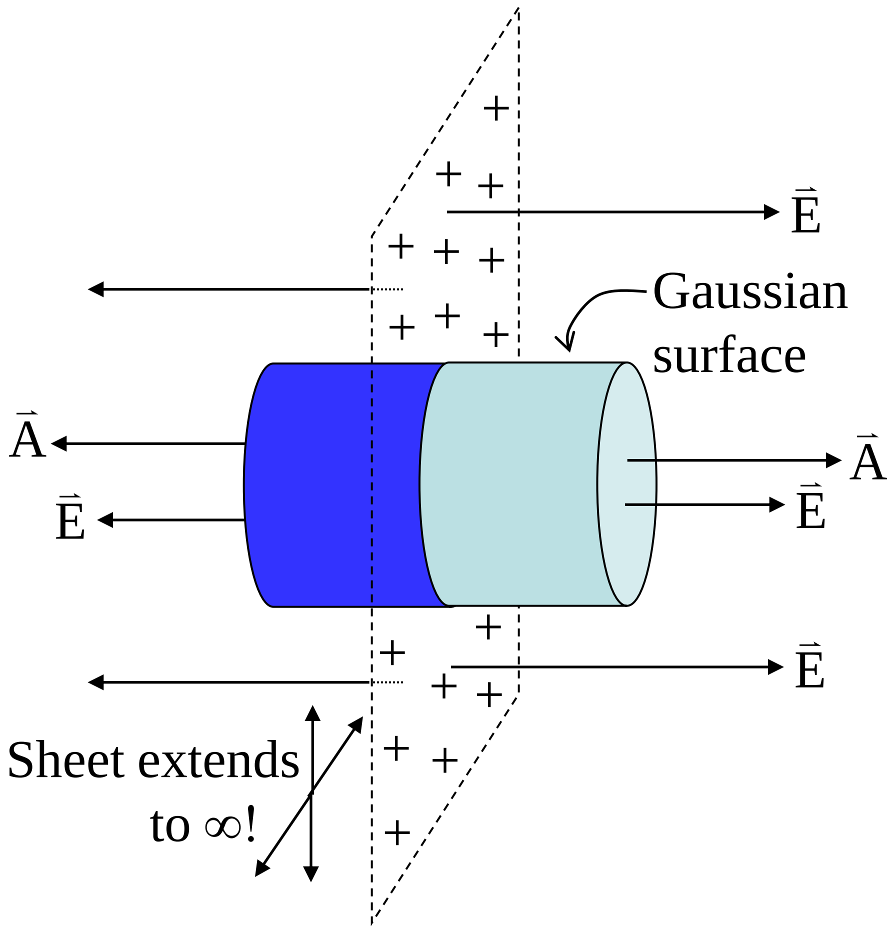
- For LH end, $\vec{E} = 0.$
- Hence:

$$EA = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{\varepsilon_0} \qquad [4.2]$$

- Field perpendicular to surface, directed out of conductor.

Field due to Infinite Charged Sheet

- Surface charge density σ.



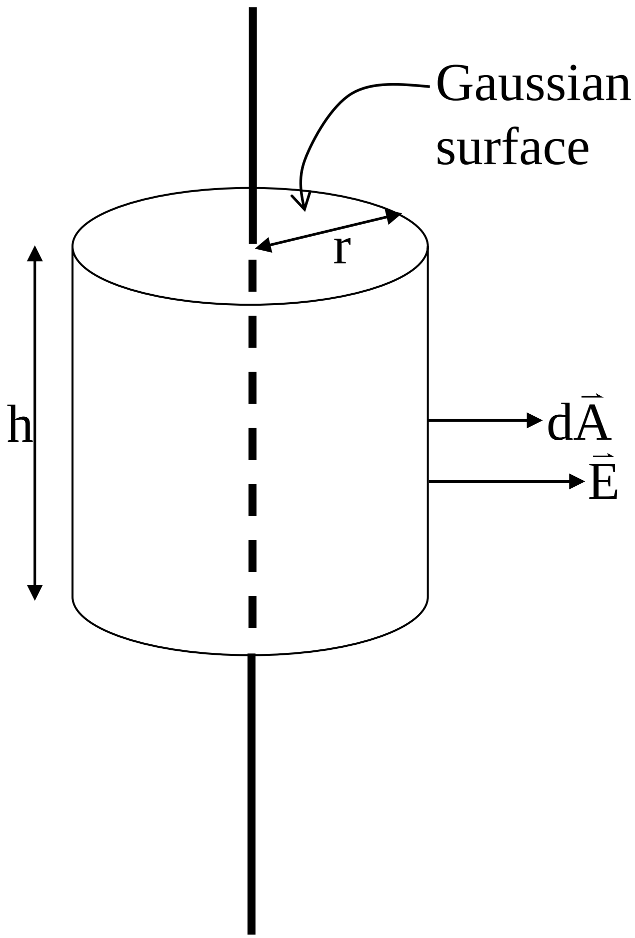
- Analysis similar to that for field outside infinite charged conductor.
- Difference is that there is a field present at the LH end of the Gaussian surface.
- Leads to result that field strength is:

$$E = \frac{1}{2}\frac{\sigma}{\varepsilon_0} \qquad [4.3]$$

- Field is directed perpendicular to sheet, towards both left and right.

Field due to Infinite Line of Charge

- Linear charge density λ (C m^{-1}).



- Applying Gauss' Law:

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda h}{\varepsilon_0}.$$

- Symmetry implies E field directed radially outwards.
- No contribution from cylinder ends (E field perpendicular to area vectors).
- E constant at given radius, so

$$\oint_S \vec{E}\cdot d\vec{A} = E\int_{Sides} dA = E(2\pi rh).$$

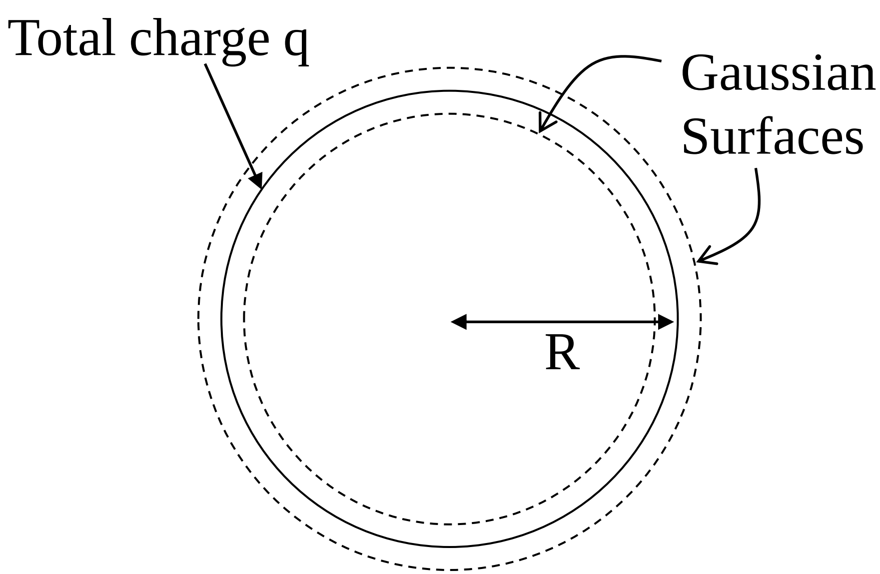
- Hence:

$$E(2\pi rh) = \frac{\lambda h}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\lambda}{2\pi r\varepsilon_0} \qquad [4.4]$$

Gauss' Law and the Shell Theorem

- What does the Shell Theorem state?
- Consider charge distributed uniformly over a spherical shell:



- Symmetry implies E field at each of Gaussian surfaces has constant magnitude and is radially directed.
- Consider surface outside shell:

$$E \oint_{S_{out}} dA = \frac{q}{\varepsilon_0}$$

$$\Rightarrow E(4\pi r^2) = \frac{q}{\varepsilon_0} \text{ or } E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.$$

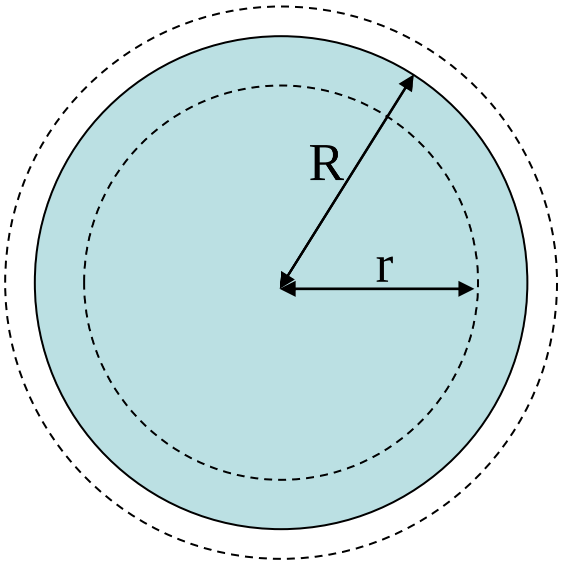
- For surface inside shell:

$$E \oint_{S_{in}} dA = 0$$

$$\Rightarrow E(4\pi r^2) = 0 \text{ or } E = 0.$$

Sphere of Charge

- Consider uniform sphere of charge:



- Total charge on sphere q.
- Outside sphere, analysis as for shell:

$$E\oint_{S_{out}} dA = \frac{q}{\varepsilon_0} \Rightarrow E(4\pi r^2) = \frac{q}{\varepsilon_0} \text{ or :}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad [4.5]$$

- Inside sphere:

$$E\oint_{S_{in}} dA = \frac{q_{enc}}{\varepsilon_0} = \frac{q(\frac{4}{3}\pi r^3 / \frac{4}{3}\pi R^3)}{\varepsilon_0}$$

$$\Rightarrow E(4\pi r^2) = \frac{qr^3}{\varepsilon_0 R^3}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3} \qquad [4.6]$$

$$= \frac{1}{4\pi\varepsilon_0} q \frac{r^3}{R^3}\frac{1}{r^2}.$$

- That is, at a radius r inside the sphere, the field is the same as that due to a point charge at the centre of the sphere, with magnitude of all the charge within the radius r.