RESONANCE AMPLIFICATION AND *T*-VIOLATION EFFECTS IN THREE-NEUTRINO OSCILLATIONS IN THE EARTH

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The matter effects in three-neutrino oscillations in a beam of neutrinos passing through the earth are discussed. An exact analytic expression for the *T*-violating asymmetry in oscillations involving the three flavour neutrinos in matter with constant density is derived and it is shown that this asymmetry can be amplified considerably by matter. In particular, the *T*-violating asymmetry in the oscillations of neutrinos which have passed through the earth can be much larger than the analogous asymmetry in the oscillations in vacuum.

There has been considerable interest in the possible effects of matter on neutrino oscillations [1–3] and they were intensively studied in the last two and a half years [#1]. It was stimulated by the pioneering work of Mikheyev and Smirnov [3] who showed that under certain conditions the presence of matter can lead to a resonance amplification of the neutrino transitions, even if these transitions are strongly suppressed in vacuum. It was also found in ref. [3] that for a large range of values of the parameters characterizing the neutrino oscillations the conditions indicated can occur in the sun and thus a substantial reduction of the flux of solar electron-neutrinos on their way from the central region to the surface of the sun is possible. In this way, matter-enhanced neutrino oscillations were shown to provide a possible very attractive solution of the solar neutrino problem [6].

Most of the studies of the matter effects in neutrino oscillations have been performed under the assumption that only two neutrinos take part in the oscillations. Three-neutrino oscillations in matter with varying density have been considered in refs. [7–10]. However, the calculations of the relevant three-neutrino transition probabilities performed so far are based on simplifying assumptions (small neutrino mixing angles in vacuum, large neutrino mass hierarchy, etc.); and the results of these calculations were used to describe the oscillations of solar neutrinos only. In this letter we present the results of a numerical integration of the system of evolution equations describing the oscillations in a beam of neutrinos passing through the earth, when three neutrinos take part in the oscillations.

The possible matter effects in the oscillations of neutrinos passing through the earth have been studied first in ref. [2]. Limiting their discussion to the case of constant density and large lepton mixing angles, the authors of ref. [2] have obtained only a rather qualitative picture of the three-neutrino oscillations in the earth. Detailed calculations of the effects of earth matter on the two-neutrino oscillations of accelerator and atmospheric neutrinos have been performed in refs. [11–14] by using more realistic distributions of the matter density ρ_E in the earth. It was found in particular, that resonance amplification of the probability of a transition between two neutrinos can take place for $\mathrm{tg}\,2\theta \gtrsim 0.1$ and $E/\Delta m^2 \gtrsim 10^3$ $\mathrm{GeV/eV^2}$, where θ is the neutrino mixing angle in vacuum, E is the neutrino energy and Δm^2 is the difference of the squares of the masses of neutrinos with definite mass in vacuum. A method of reconstruction of the density distribution in the earth, ρ_E, by measuring the dependence of the probability $P(\nu_e \to \nu_e)$ that the electron-neutrino will not change into a different type of neutrino while crossing the earth on the neutrino energy in experiments with ac-

[#1] See, e.g. the review articles [4,5] where extensive lists of references are given.

celerator neutrinos was proposed in ref. [11] [#2]. Although the method of ρ_E reconstruction suggested in ref. [11] has some common traits with the method of neutrino tomography [15] it can be more sensitive to the matter density distribution in the earth at lower neutrino energies.

In the present letter we study the effects of the earth matter on three-neutrino oscillations by solving numerically the corresponding system of neutrino evolution equations without making any assumptions simplifying the equations. We are especially interested in effects which are specific for the oscillations involving three (or more) neutrinos. In particular, we investigate in detail the dependence of the probabilities of the three-neutrino oscillations in the earth on the *CP*- (*T*-) violating phase which might be present in the lepton mixing matrix.

We shall consider oscillations involving the three flavour neutrinos ν_e, ν_μ and ν_τ. With minor modifications our results will be valid for oscillations in the earth in which the antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$ and $\bar{\nu}_\tau$ take part. Having in mind possible applications for accelerator and/or atmospheric neutrinos, we shall be interested in the transitions $\nu_e \to \nu_\ell$, $\ell = \mu, \tau$ and $\nu_\mu \to \nu_{\ell'}$, $\ell' = e, \tau$. The system of evolution equations describing the neutrino transitions can be written in the form

$$i \frac{d}{dt} a_\ell(t) = \sum_{\ell' = e,\mu,\tau} M_{\ell\ell'}(t) a_{\ell'}(t), \quad \ell = e, \mu, \tau. \qquad (1)$$

Here $a_\ell(t)$ is the amplitude of the probability to find neutrino ν_ℓ at time t and $M(t)$ is the neutrino evolution matrix in matter, which we take as [8]

$$M_{\ell\ell'}(t) = \frac{1}{2E} \left[\sum_{j=1}^{3} U_{\ell j} m_j^2 U^+_{j\ell'} + A(t) \delta_{\ell\ell'} \delta_{\ell e} - \delta_{\ell\ell'} \left(\sum_{j=1}^{3} |U_{ej}|^2 m_j^2 + A(t) \right) \right]. \qquad (2)$$

In eq. (2) $E = |\boldsymbol{p}|$, where $\boldsymbol{p}$ is the neutrino momentum, U is a 3×3 unitary matrix – the neutrino mixing matrix in vacuum

$$|\nu_\ell(\boldsymbol{p})\rangle = \sum_{j=1}^{3} U_{\ell j} |\nu_j(\boldsymbol{p}, m_j)\rangle , \qquad (3)$$

$|\nu_\ell(\boldsymbol{p})\rangle$ being the state of neutrino ν_ℓ, $\ell = e, \mu, \tau$, with momentum $\boldsymbol{p}$ and $|\nu_j(\boldsymbol{p}, m_j)\rangle$ being the state of neutrino ν_j with momentum $\boldsymbol{p}$ and definite mass m_j in vacuum [#3],

$$A(t) = 2E\sqrt{2} G_F N_e(t) , \qquad (4)$$

where [1, 2, 16] $N_e(t)$ is the electron number density.

The amplitude $a_\ell(t)$ of the probability to find the antineutrino $\bar{\nu}_\ell$, $\ell = e, \mu, \tau$, at time t in the case of oscillations involving antineutrinos satisfies a system of evolution equations analogous to (1). The corresponding antineutrino evolution matrix $\bar{M}(t)$ can be obtained [8] from (2) by making the formal change $U \to U^*$ and $A(t) \to -A(t)$.

We shall denote the lightest and the heaviest vacuum mass eigenstate neutrinos by ν_1 and ν_3:

$$m_1 < m_2 < m_3 . \qquad (5)$$

It proves to be convenient to use the following parametrization of the mixing matrix in vacuum, U, in this case [7,8]:

$$U = O^{23}(\varphi_{23}) O^{33}(\delta) O^{13}(\varphi_{13}) O^{12}(\varphi_{12}) . \qquad (6)$$

Here

$$O^{12}(\varphi_{12}) = \begin{pmatrix} \cos\varphi_{12} & \sin\varphi_{12} & 0 \\ -\sin\varphi_{12} & \cos\varphi_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$O^{13}(\varphi_{13}) = \begin{pmatrix} \cos\varphi_{13} & 0 & \sin\varphi_{13} \\ 0 & 1 & 0 \\ -\sin\varphi_{13} & 0 & \cos\varphi_{13} \end{pmatrix},$$

$$O^{23}(\varphi_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_{23} & \sin\varphi_{23} \\ 0 & -\sin\varphi_{23} & \cos\varphi_{23} \end{pmatrix},$$

$$O^{33}(\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}, \qquad (7)$$

where φ_{12}, φ_{13} and φ_{23} are neutrino mixing angles in vacuum and δ is a *CP*- (*T*-) violating phase [#4]. Without loss of generality, the angles φ_{12}, φ_{13} and φ_{23} can

[#2] This method can be used only if the parameters Δm^2 and θ are known.

[#3] The vacuum mass eigenstate neutrinos ν_j are assumed to be stable and relativistic: $E_j = \sqrt{p^2 + m_j^2} \cong E + m_j^2/2E$.

[#4] As was shown in refs. [17,18], the three-neutrino oscillation probabilities of interest can depend only on one *CP*- (*T*-) violating phase (the so-called Dirac phase [5]) present in the lepton mixing matrix in vacuum.

be chosen to vary in the interval $[0, \pi/2]$, while the phase δ can take values in the interval $[0, 2\pi]$.

Being a hermitian matrix, the neutrino evolution matrix $M(t)$ can be diagonalized for any given t with the help of an unitary matrix $U^{m}(t)$:

$$U^{m\dagger}(t)M(t)U^{m}(t)=M^{d}(t)\,, \tag{8}$$

where $M^{d}(t)$ is a real diagonal matrix whose elements $M_j^2/2E$, $j=1, 2, 3$, represent the eigenvalues of $M(t)$ at time t.

For $\Delta m_{21}^2>0$ and $\Delta m_{31}^2>0$, there can be two resonances (one resonance) in the transitions of electron-neutrinos (muon-neutrinos) of a given energy when they traverse the earth [7,8]. The resonances are most pronounced in the case of small vacuum mixing angles [7], $\sin\varphi_{12}\ll 1$, $\sin\varphi_{13}\ll 1$, and when they are sufficiently separated [8] (i.e. if $\Delta m_{31}^2-\Delta m_{21}^2\gg\Delta m_{31}^2\sin\varphi_{13}+\Delta m_{21}^2\sin\varphi_2$). Under these conditions, they occur at electron number densities

$$N_{L}^{res}\cong\Delta m_{21}^2\cos 2\varphi_{12}/2E\sqrt{2}G_F\,,$$
$$N_{H}^{res}\cong\Delta m_{31}^2\cos 2\varphi_{13}/2E\sqrt{2}G_F\,. \tag{9}$$

For $\sin\varphi_{23}\ll 1$, the resonance at the lower density (N_L^{res}) takes place in the $\nu_{e(\mu)}\to\nu_{\mu(e)}$ transition while the resonance at the higher density (N_H^{res}) occurs in the $\nu_e\to\nu_\tau$ transition. In general, the resonances occur at densities at which the differences $M_2^2(t)-M_1^2(t)$ and $M_3^2(t)-M_2^2(t)$ take minimal values [4]. As can be easily shown, the eigenvalues of $M(t)$ do not depend on the *CP*- (*T*-) violating phase δ in the case of the parametrization (6) of U. Therefore, the existence of the resonances as well as the values of the electron number density at which they can occur are independent of the value of δ. However, the probabilities $P(\nu_{e(\mu)}\to\nu_\ell)$ of the transitions of interest $\nu_{e(\mu)}\to\nu_\ell$, $\ell=e, \mu, \tau$, can depend in a nontrivial way on δ.

In our analysis of the effects of earth matter on the three-neutrino oscillations we have used the density distribution in the earth $\rho_E(r)$ (r is the distance from the centre of the earth) predicted by the preliminary reference earth model [19] (PREM). According to it $\rho_E(r)$ increases towards the center of the earth, where it takes its maximal value of 13.1 g/cm^3. There are basically seven regions in which $\rho_E(r)$ changes continuously. The change of $\rho_E(r)$ on the borders of these regions is described by step functions. These jumps in the value of $\rho_E(r)$ are rather small (less than 0.6 g/cm^3) except on the border between the (lower) mantle and (outer) core where $\rho_E(r)$ changes from 5.6 g/cm^3 to 9.9 g/cm^3. According to the PREM the earth's radius is 6371 km and the indicated large discontinuity of $\rho_E(r)$ takes place at a distance of 3480 km from the centre of the earth. It should also be noted that the first four regions of continuous $\rho_E(r)$ variation are rather narrow (they extend together from the surface to $r\simeq 5700$ km) and the density changes little (from 2.6 g/cm^3 to 4.0 g/cm^3) in them. Our calculations were performed assuming that the earth matter is isotopically symmetric, i.e., that in any region of the earth $N_e^E=\frac{1}{2}N_N$, where N_N is the nucleon number density, $N_N=\rho_E N_A$, N_A being Avogadro's number. Let us add that the specific features of the oscillations of neutrinos passing through the earth can be understood qualitatively (and quantitatively in many cases) if one assumes that the earth consists of two regions with different constant densities: a mantle with density $\bar{\rho}_E^M\cong 4$ g/cm^3 which extends in the radial direction from the surface to $\Delta r_M\cong 2890$ km and a core with density $\bar{\rho}_E^C\cong 12$ g/cm^3 and diameter $\Delta r_C\cong 6960$ km (two-layer model).

The results of the calculations show that for neutrinos crossing the earth along its diameter matter can enhance substantially the neutrino transitions if 10^3 GeV/eV$^2\lesssim E/\Delta m_{21(31)}^2\lesssim 5\times 10^4$ GeV/eV^2 and $\sin\varphi_{12}\gtrsim 0.05$ and/or $\sin\varphi_{13}\gtrsim 0.05$, independently of the values of φ_{23} and δ. The magnitude of the indicated interval of values of $E/\Delta m_{21(31)}^2$ is determined by the magnitude of the interval of values of the electron number density met in the earth. The enhancement typically shows up in the dependence of the probability of a given transition on the neutrino energy as an irregular sequence of two–three well-pronounced local maxima with different heights. For $E/\Delta m_{21(31)}^2\gg 5\times 10^4$ GeV/eV^2 the resonance densities ($N_{H,L}^{res}$) are much smaller than the electron number density in the earth $N_e^E(r)$, ($N_e^E(r)=\frac{1}{2}\rho_E(r)N_A$, 1.0 g/cm$^3\lesssim\rho_E(r)\lesssim 13.1$ g/cm^3), and the earth matter suppresses the neutrino oscillations even in the case of large $\sin 2\varphi_{12}$ and $\sin 2\varphi_{13}$. For $E/\Delta m_{21(31)}^2\ll 10^3$ GeV/eV^2 and $\varphi_{12,13}$ not close to $\pi/4$, the resonance densities exceed $N_e^E(r)$ and neutrinos oscillate as in vacuum. Further, if $\sin\varphi_{12}<0.05$ and $\sin\varphi_{13}<0.05$, the neutrino oscillation lengths in matter at the resonances

$$L_L^{res} = L_{21}^V / |\sin 2\varphi_{12}|, \quad L_H^{res} = L_{31}^V / |\sin 2\varphi_{13}|, \tag{10}$$

where $L_{21,31}^V = 4\pi E/\Delta m_{21,31}^2$ are the corresponding oscillation lengths in vacuum, are much larger than the diameter of the earth and large amplitude oscillations cannot develop.

As an illustration of some of these general features we show in fig. 1 the dependence of the probabilities $P(\nu_e \to \nu_e)$, $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ to find the neutrinos ν_e, ν_μ and ν_τ at the surface of the earth in a beam of electron-neutrinos which crossed the earth along its diameter on $E/\Delta m_{21}^2$ for $\sin\varphi_{12}=0.25$, $\sin\varphi_{13}=0.1$, $\sin\varphi_{23}=0.3$, $\delta=\pi/3$ and $\Delta m_{31}^2 = 10\Delta m_{21}^2$. In order to understand qualitatively the form of $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$, it is sufficient to use the two-layer model of the density distribution in the earth and the fact that for the values of $\sin\varphi_{12}$, $\sin\varphi_{13}$ and $\Delta m_{31}^2/\Delta m_{21}^2$ chosen, the $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_\tau$ transitions are described essentially by two-neutrino transition probabilities with vacuum oscillation parameters [8,9] $\sin 2\varphi_{12}$, Δm_{21}^2 and $\sin 2\varphi_{13}$, Δm_{31}^2, respectively [#5]. Consider first $P(\nu_e \to \nu_\tau)$ ($E/\Delta m_{21}^2 = 10E/\Delta m_{31}^2$). The local maximum at $E/\Delta m_{21}^2 \cong 3.6 \times 10^4$ GeV/eV^2 corresponds to a resonance in the mantle ($N_H^{res} \simeq 1.75(\mathrm{g/cm^3})N_A$) where the ν_e–ν_τ mixing angle $\varphi_{13}^m \cong \pi/4$ and $\sin 2\varphi_{13}^m \cong 1$. However, the oscillations cannot develop at this value of $E/\Delta m_{21}^2$ and $P(\nu_e \to \nu_\tau) \simeq 0.08$ because the distances travelled by the neutrinos in the mantle ($\Delta r_M \cong 2890$ km) are approximately $2\pi \cdot 2.5$ times smaller than the oscillation length at resonance ($L_H^{res} \cong 2\pi \cdot 7.5 \times 10^3$ km), and because the matter in the core suppresses the ν_e–ν_τ oscillations. The second maximum of $P(\nu_e \to \nu_\tau)$ at $E/\Delta m_{21}^2 \cong 1.9 \times 10^4$ GeV/eV^2 corresponds to a resonance density $N_H^{res} = 3.3$ (g/cm^3)N_A, which exceeds $N_e^E(r)$ somewhat in the mantle but is smaller than $N_e^E(r)$ in the core. Since N_H^{res} is close to the values of the electron number density in the mantle and the oscillation length in the mantle is not large ($\sim 2\pi \cdot 1800$ km), the $\nu_e \to \nu_\tau$ transitions are amplified in the mantle, but not in the core. Clearly, the amplification cannot be maximal ($P(\nu_e \to \nu_\tau) \cong 0.55$) because nowhere in the mantle is the density equal to the resonance density. The third distinct maximum of $P(\nu_e \to \nu_\tau)$ at $E/\Delta m_{21}^2 \cong 1.2 \times 10^4$ GeV/eV^2 results from a maximal enhancement of the $\nu_e \to \nu_\tau$ transition in the core of the earth ($N_H^{res} = 5.5(\mathrm{g/cm^3})N_A$). For 10^3 GeV/eV$^2 \lesssim E/\Delta m_{21}^2 \lesssim 8 \times 10^3$ GeV/eV^2 the effects of matter on $P(\nu_e \to \nu_\tau)$ are noticeable but not

[#5] The relevant two-neutrino transition probabilities must be multiplied by the factor $\cos^2\varphi_{23}$ which in this case accounts for the participation of the third neutrino in the oscillations.

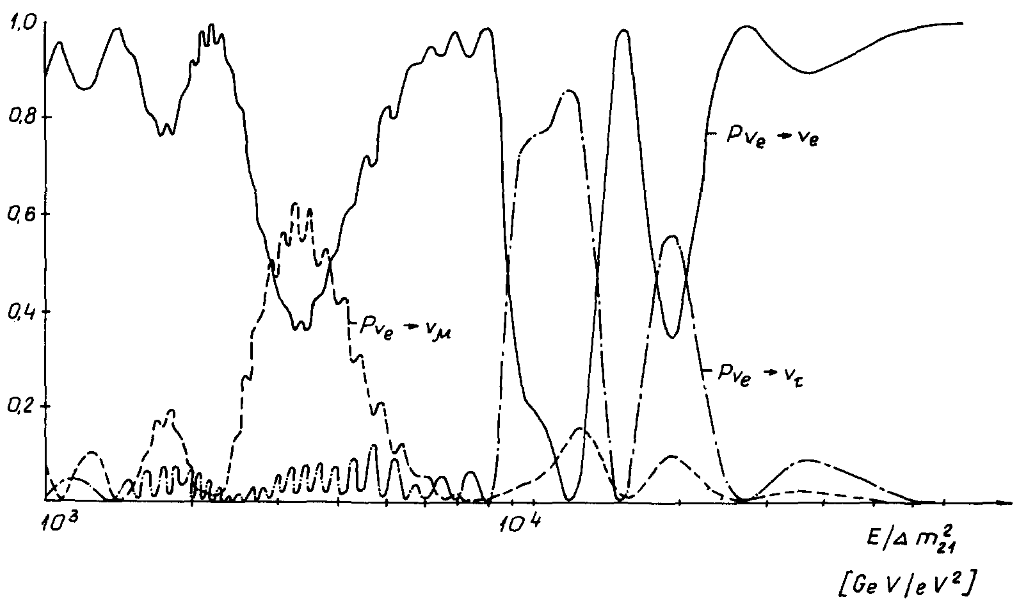

Fig. 1. The dependence of the three-neutrino oscillations probabilities $P(\nu_e \to \nu_e)$, $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ for neutrinos which crossed the earth along its diameter on $E/\Delta m_{21}^2$ for $\sin\varphi_{12}=0.25$, $\sin\varphi_{13}=0.1$, $\sin\varphi_{23}=0.3$, $\delta=\frac{1}{3}\pi$ and $\Delta m_{31}^2 = 10\Delta m_{21}^2$.

dramatic ($P(\nu_e \to \nu_\tau) \lesssim 0.1$). Let us add that the precise position of the maxima and minima of $P(\nu_e \to \nu_\tau)$ are determined by the relative magnitudes of the distances travelled by the neutrinos and the oscillation lengths in the mantle and in the core. The latter depend on the oscillation length in vacuum (L^{V}_{31}), the ratio of the corresponding density and the resonance density and on the mixing angle in vacuum (i.e., $\sin 2\varphi_{13}$).

One can analyze the dependence of $P(\nu_e \to \nu_\mu)$ on $E/\Delta m^2_{21}$ in a similar way. For instance, the most pronounced maximum of $P(\nu_e \to \nu_\mu)$ at $E/\Delta m^2_{21} \cong 3.2 \times 10^3\ \mathrm{GeV/eV^2}$ (see fig. 1) corresponds to a resonance in the mantle ($N^{\mathrm{res}}_{\mathrm{L}} \cong 2.0(\mathrm{g/cm^3})N_{\mathrm{A}}$). As we have seen, resonance amplification of the ν_e–ν_τ mixing (i.e., of $\sin 2\varphi^{\mathrm{m}}_{13}$) occurs rougly at the same resonance density. However, since $\sin 2\varphi_{12} = 2.5 \sin 2\varphi_{13}$, $L^{\mathrm{res}}_{\mathrm{L}} \cong 0.4 L^{\mathrm{res}}_{\mathrm{H}} \cong 2\pi \times 3000$ km (see eq. (10)) and the ν_e–ν_μ oscillations can develop in the mantle (while the ν_e–ν_τ oscillations cannot). The $\nu_e \to \nu_\mu$ transition is not complete ($P(\nu_e \to \nu_\mu) \cong 0.63$) because $2\pi \cdot \Delta r_{\mathrm{M}} \cong L^{\mathrm{res}}_{\mathrm{L}}$.

Let us note that the relative position of the minima and maxima of $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ varies with $\sin\varphi_{12}$, Δm^2_{21}, $\sin\varphi_{13}$ and Δm^2_{31}. However, if $\sin\varphi_{12} \cong \sin\varphi_{13} \ll 1$, $\sin\varphi_{23} \ll 1$ (say, $\sin\varphi_{12,13,23} \lesssim 0.3$) and the resonances in the $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_\tau$ transitions are sufficiently separated ($\Delta m^2_{31} - \Delta m^2_{21} \gg \Delta m^2_{31} \sin\varphi_{13} + \Delta m^2_{21} \sin\varphi_{12}$) we should have [8,9]

$$P(\nu_e \to \nu_\mu)\,|_{E/\Delta m^2_{21}} \cong P(\nu_e \to \nu_\tau)\,|_{(E/\Delta m^2_{21})\cdot(\Delta m^2_{31}/\Delta m^2_{21})} \,.$$

The dependence of $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ on $E/\Delta m^2_{21,31}$ is reflected in the dependence of $P(\nu_e \to \nu_e)$ on $E/\Delta m^2_{21,31}$ ($P(\nu_e \to \nu_e) = 1 - P(\nu_e \to \nu_\mu) - P(\nu_e \to \nu_\tau)$). Therefore, $P(\nu_e \to \nu_e)$ is sensitive to the matter effects in each of the ν_e transitions ($\nu_e \to \nu_\mu$, $\nu_e \to \nu_\tau$, ...). As a consequence, these effects in $P(\nu_e \to \nu_e)$ can be clearly distinguishable in an interval of values of the neutrino energy which can extend over 2–3 (or more) orders of magnitude. This implies also that $P(\nu_e \to \nu_e)$ is, in general, more sensitive to the earth matter effects in the oscillations of neutrinos than $P(\nu_\mu \to \nu_\mu)$ since essentially only one type of muon-neutrino transitions, namely $\nu_\mu \to \nu_e$, can be amplified in matter [7–10].

How will the dependence of, e.g. $P(\nu_e \to \nu_\tau)$ on $E/\Delta m^2_{21}$ shown in fig. 1 change if the beam of electron-neutrinos does not cross the earth along its diameter? As the path of the beam moves further and further away from the diameter, the maximum at $E/\Delta m^2_{21} \cong 1.2 \times 10^4\ \mathrm{GeV/eV^2}$ (resonance in the core) will decrease and gradually disappear since the neutrino path in the core becomes shorter and shorter. The maximum at $E/\Delta m^2_{21} \cong 3.6 \times 10^4\ \mathrm{GeV/eV^2}$ (resonance in the mantle) will first increase $P(\nu_e \to \nu_\tau)$ reaching the value 0.4 at it) together with the length of the neutrino path in the mantle; as the latter begins to diminish, the maximum will begin to decrease and will gradually also disappear. Finally, the maximum at $E/\Delta m^2_{21} \cong 1.9 \times 10^4\ \mathrm{GeV/eV^2}$ will also first increase slightly; this will be followed by a decrease.

Let us next discuss the role of the *CP*- (*T*-) violating phase δ in three-neutrino oscillations in matter. The presence of the phase $\delta \neq 0$, π in the lepton mixing matrix in vacuum U (eq. (6)) leads to both *CP*-violation and *T*-violation effects in neutrino oscillations [5] i.e., to $P(\nu_\ell \to \nu_{\ell'}) \neq P(\bar\nu_\ell \to \bar\nu_{\ell'})$ and $P(\overset{(-)}{\nu}_\ell \to \overset{(-)}{\nu}_{\ell'}) \neq P(\overset{(-)}{\nu}_{\ell'} \to \overset{(-)}{\nu}_\ell)$, $\ell \neq \ell' = \mathrm{e}, \mu, \tau, \ldots$, respectively. In the case of oscillations taking place in matter, the difference between $P(\nu_\ell \to \nu_{\ell'})$ and $P(\bar\nu_\ell \to \bar\nu_{\ell'})$ $\ell, \ell' = \mathrm{e}, \mu, \tau, \ldots$, can be generated by the matter itself even if $\delta = 0$ since the matter surrounding us is not symmetric with respect to charge (*C*-) conjugation and, therefore, the interaction of the neutrinos with matter is neither *CP*- nor *CPT*-invariant [18]. In such a way only the observation of a nonzero *T*-violating asymmetry in the neutrino oscillations in matter

$$\overset{(-)}{A}{}^{(\ell',\ell)}_{T} = P(\overset{(-)}{\nu}_\ell \to \overset{(-)}{\nu}_{\ell'}) - P(\overset{(-)}{\nu}_{\ell'}$$

$$\ell \neq \ell' = e, \mu, \tau, \ldots \tag{11}$$

can be unambiguously associated with the existence of a nontrivial *CP*- (*T*-) violating phase in the lepton mixing matrix. It follows from the probability conservation,

$$\sum_{\ell' = \mathrm{e},\mu,\tau} P(\overset{(-)}{\nu}_\ell \to \overset{(-)}{\nu}_{\ell'}) = 1,$$

$$\sum_{\ell' = \mathrm{e},\mu,\tau} P(\overset{(-)}{\nu}_{\ell'} \to \overset{(-)}{\nu}_\ell) = 1\,, \tag{12}$$

that in the case of oscillations involving the three flavour neutrinos (antineutrinos) $\overset{(-)}{\nu}_e$, $\overset{(-)}{\nu}_\mu$ and $\overset{(-)}{\nu}_\tau$ one only has [#6]

[#6] We thank S. Toshev for a discussion on this point.

$$^{(}\bar{A}^{)(\mu;e)}_T = {}^{(}\bar{A}^{)(\tau;\mu)}_T = {}^{(}\bar{A}^{)(e;\tau)}_T \equiv {}^{(}\bar{A}^{)}_T \,. \qquad (13)$$

If the oscillations take place in vacuum, *CPT*-invariance holds and we get using (3) and the expressions for $P({}^{(}\bar{\nu}^{)}_{\ell} \to {}^{(}\bar{\nu}^{)}_{\ell'})$ given in ref. [5]

$$A^{\mathrm{V}}_T = -\bar{A}^{\mathrm{V}}_T$$
$$= 4\mathscr{J}^{\mathrm{V}}[\sin(\Delta m^2_{12}/2E)R + \sin(\Delta m^2_{31}/2E)R + \sin(\Delta m^2_{23}/2E)R] \qquad (14)$$

where R is the distance travelled by the neutrinos and $\mathscr{J}^{\mathrm{V}}$ is a rephasing invariant [20] formed by the elements of the lepton mixing matrix in vacuum [#7]

$$\mathscr{J}^{\mathrm{V}} = \mathrm{Im}(U_{e2}U_{\mu 3}U^*_{e3}U^*_{\mu 2})$$
$$= -\mathrm{Im}(U_{e2}U_{\tau 3}U^*_{\tau 2}U^*_{e3}) = \dots \,. \qquad (15)$$

For three lepton families the invariant $\mathscr{J}^{\mathrm{V}}$ is unique in the sense that the *T*-violation effects in neutrino oscillations should be proportional to it. In the parametrization (6) we are using

$$\mathscr{J}^{\mathrm{V}} = \tfrac{1}{8}\sin\delta\cdot\sin 2\varphi_{12}\cdot\sin 2\varphi_{13}\cdot\sin 2\varphi_{23}\cdot\cos\varphi_{13}. \qquad (16)$$

It follows from eqs. (14) and (16) that ${}^{(}\bar{A}^{)\mathrm{V}}_T = 0$ and *T*- (*CP*-) invariance holds in the neutrino oscillations in vacuum if one of the following conditions is fulfilled: (i) $\delta = 0$ or π; (ii) two of the vacuum mass eigenstate neutrinos ν_j, $j = 1, 2, 3$ are mass-degenerate; (iii) one of the mixing angles in vacuum is equal to 0 or $\pi/2$. These conditions are analogous to the conditions of *T*- (*CP*-) invariance in the quark sector of the standard model with three families of quarks and leptons [20].

If neutrino oscillations take place in matter with constant density, the matrices $M(t)$, $U^{\mathrm{m}}(t)$ and $M^{\mathrm{d}}(t)$ (see eqs. (2) and (8) will be independent of t; U^{m} will be the lepton mixing matrix in matter, while the eigenvalues of M (more precisely, $(E + M^2_j/2E)$) will represent the energies of the mass eigenstate neutrinos in matter. Assuming that U^{m} and M^2_j are known, it is not difficult to derive in this case an expression for the *T*-violating asymmetry analogous to (14):

$$A^{\mathrm{m}}_T = 4\mathscr{J}^{\mathrm{m}}[\sin(\Delta M^2_{12}/2E)R + \sin(\Delta M^2_{31}/2E)R + \sin(\Delta M^2_{23}/2E)R] \,. \qquad (17)$$

[#7] The quantity $\mathscr{J}^{\mathrm{V}}$ (as like the neutrino oscillation probabilities [17,18,5] is invariant with respect to multiplication of the rows and the columns of the lepton mixing matrix U by phase factors.

Here $\Delta M^2_{ij} = M^2_i - M^2_j$ and $\mathscr{J}^{\mathrm{m}}$ is the rephasing invariant of the neutrino oscillations in matter with constant density,

$$\mathscr{J}^{\mathrm{m}} = \mathrm{Im}(U^{\mathrm{m}}_{e2}U^{\mathrm{m}}_{\mu 3}U^{\mathrm{m}*}_{e3}U^{\mathrm{m}*}_{\mu 2})$$
$$= -\mathrm{Im}(U^{\mathrm{m}}_{e2}U^{\mathrm{m}}_{\tau 3}U^{\mathrm{m}*}_{e3}U^{\mathrm{m}*}_{\tau 2}) = \dots \,. \qquad (18)$$

Up to a diagonal unitary matrix (ϕ) containing unobservable phases U^{m} can be parametrized as U:

$$U^{\mathrm{m}} = \phi O^{23}(\varphi^{\mathrm{m}}_{23})O^{33}(\delta^{\mathrm{m}})O^{13}(\varphi^{\mathrm{m}}_{13})O^{12}(\varphi^{\mathrm{m}}_{12}), \qquad (19)$$

where $\varphi^{\mathrm{m}}_{12}$, $\varphi^{\mathrm{m}}_{13}$ and $\varphi^{\mathrm{m}}_{23}$ are the neutrino mixing angles and δ^{m} is the *T*- (*CP*-) violating phase in matter with constant density, and the matrices O^{12}, ..., O^{33} are defined in eq. (7). From (18) and (19) we get

$$\mathscr{J}^{\mathrm{m}} = \tfrac{1}{8}\sin\delta^{\mathrm{m}}\sin 2\varphi^{\mathrm{m}}_{12}\sin 2\varphi^{\mathrm{m}}_{13}\sin 2\varphi^{\mathrm{m}}_{23}\cos\varphi^{\mathrm{m}}_{13} \,. \qquad (20)$$

The quantities M^2_j and U^{m}, and consequently $\mathscr{J}^{\mathrm{m}}$, are functions of the masses m_j, the elements of the mixing matrix in vacuum U and N_e. Exact analytic expressions for M^2_j, $j = 1, 2, 3$ in terms of m^2_j, $U_{\ell j}$ and N_e have been given in ref. [2] [#8]. We have derived analogous expressions for the elements of the mixing matrix in matter U^{m} (as well as for $\varphi^{\mathrm{m}}_{12}$, $\varphi^{\mathrm{m}}_{13}$, $\varphi^{\mathrm{m}}_{23}$ and δ^{m}). This enables us to find $\mathscr{J}^{\mathrm{m}}$:

$$\mathscr{J}^{\mathrm{m}} = \mathscr{J}^{\mathrm{V}}F(\varphi_{12}, \varphi_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A) \,, \qquad (21)$$

where F is a function of φ_{12}, φ_{13}, Δm^2_{21}, Δm^2_{31} and A (see eq. (4)) but does not depend on φ_{23} and δ. The function F can be written as

$$F = (F_1/F_2F_3)(m^2_1 - M^2_2)(m^2_3 - M^2_2) \times (m^2_1 - M^2_3)(m^2_2 - M^2_3) \,, \qquad (22)$$

where

$$F_1 = (m^2_1 - M^2_3)(m^2_2 - M^2_3)(m^2_1 - M^2_2)(m^2_3 - M^2_2)$$
$$+ A\{(m^2_1 - M^2_3)(m^2_2 - M^2_3) \times [(m^2_3 - M^2_2) - \Delta m^2_{31}U^2_{e3}]$$
$$+ (m^2_1 - M^2_2)(m^2_3 - M^2_2) \times [(m^2_2 - M^2_3) - \Delta m^2_{21}U^2_{e2}]\}$$
$$+ A^2\{U^2_{e1}(m^2_3 - M^2_2)(m^2_2 - M^2_3) + U^2_{e2}(m^2_3 - M^2_2)(m^2_1 - M^2_3) + U^2_{e3}(m^2_1 - M^2_2)(m^2_2 - M^2_3)\}, \qquad (23)$$

[#8] Note that the correct expressions for M^2_j can be obtained from those given in ref. [2] by making the formal change $N_e \to -N_e$.

$$F_2 = U_{e1}^2 (m_1^2 - M_2^2 + A)^2 (m_3^2 - M_2^2)^2 + U_{e2}^2 (m_1^2 - M_2^2)^2 (m_3^2 - M_2^2)^2 + U_{e3}^2 (m_1^2 - M_2^2)^2 (m_3^2 - M_2^2 + A)^2 - U_{e1}^2 U_{e3}^2 A^2 (\Delta m_{31}^2)^2 , \quad (24)$$

$$F_3 = F_2 |_{m_2^2 \leftrightarrow m_3^2, M_2^2 \leftrightarrow M_3^2, U_{e2}^2 \leftrightarrow U_{e3}^2} . \quad (25)$$

It follows from eqs. (22)–(25) that in the case of oscillations in vacuum, i.e. for $N_e = 0$ ($A = 0$), one has

$$F(\varphi_{12}, \varphi_{13}, \Delta m_{21}^2, \Delta m_{31}^2, 0) = 1 , \quad (26)$$

and that F is symmetric with respect to the interchange of m_2^2 and m_3^2, M_2^2 and M_3^2 and U_{e2}^2 and U_{e3}^2.

Let us analyze eqs. (17), (20) and (21) briefly. The dependence of ΔM_{ij}^2 on Δm_{ij}^2 (see ref. [2]) and eqs. (17) and (21) imply that A_T^m can be nonzero only if $A_T^V \neq 0$. In principle, we can have $|A_T^m| > |A_T^V|$. This possibility can be realized, for instance, if $\sin\varphi_{12,13} \ll 1$, while both $\sin 2\varphi_{12}^m$ and $\sin 2\varphi_{13}^m$ get amplified in matter. The latter can take place only if Δm_{21}^2 and Δm_{31}^2 do not differ substantially (say, $\Delta m_{31}^2 \sim (2\text{–}3)\Delta m_{21}^2$). Since φ_{23} is changed little by matter [8], ($\varphi_{23}^m \simeq \varphi_{23}$), $|A_T^m|$ can only be relatively large in this case if $|\sin 2\varphi_{23}|$ is not small.

For neutrinos crossing the earth along its diameter, the indicated possibility is illustrated in fig. 2, where the dependence of the normalized T-violating symmetry $I_T = [P(\nu_e \to \nu_\mu) - P(\nu_\mu \to \nu_e)] / [P(\nu_e \to \nu_\mu) + P(\nu_\mu \to \nu_e)]$ and the probabilities $P(\nu_e \to \nu_\mu)$ and

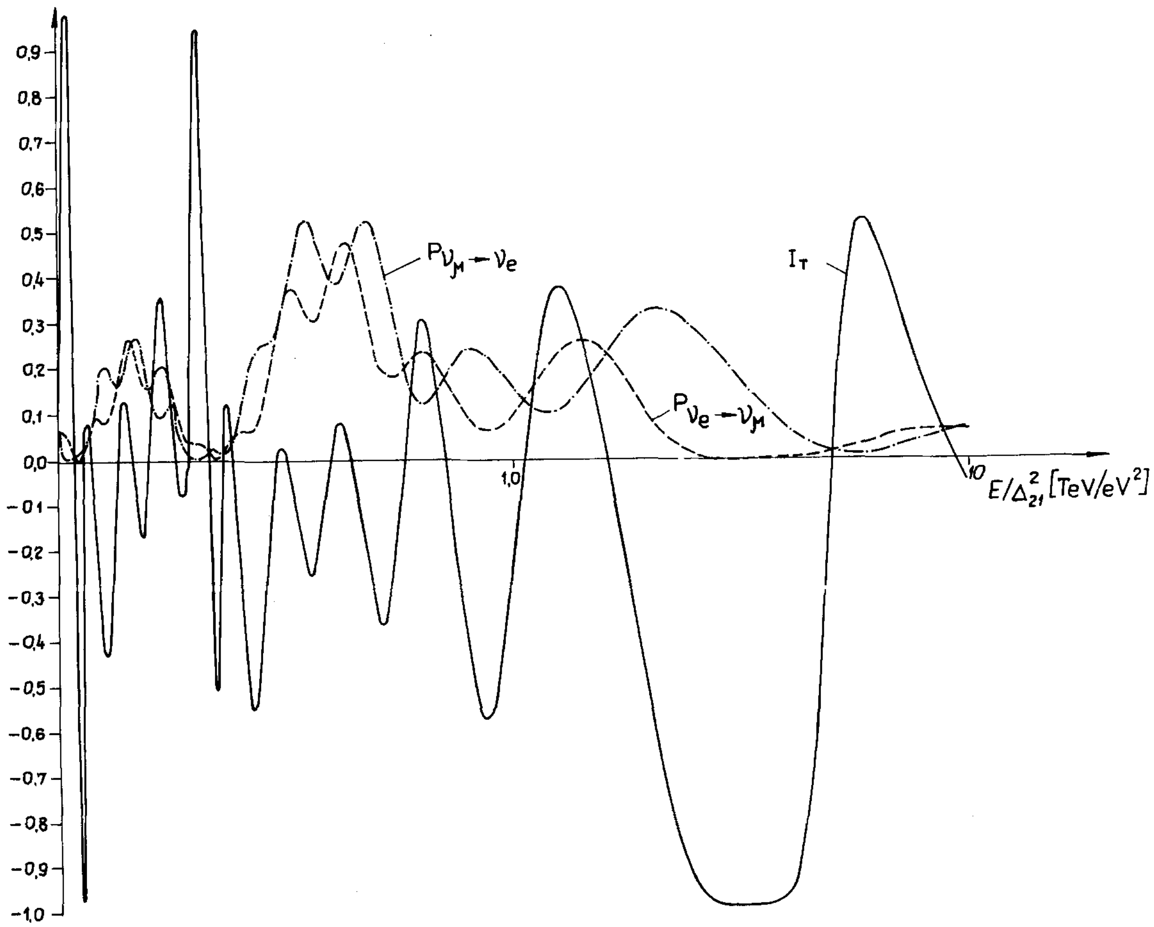

Fig. 2. The dependence of the T-violating asymmetry I_T and of the probabilities $P(\nu_e \to \nu_\mu)$ and $P(\nu_\mu \to \nu_e)$ on $E/\Delta m_{21}^2$ for $\sin\varphi_{12} = 0.25$, $\sin\varphi_{13} = 0.2$, $\sin\varphi_{23} = 0.3$, $\delta = \frac{1}{3}\pi$ and $\Delta m_{31}^2 = 3\Delta m_{21}^2$.

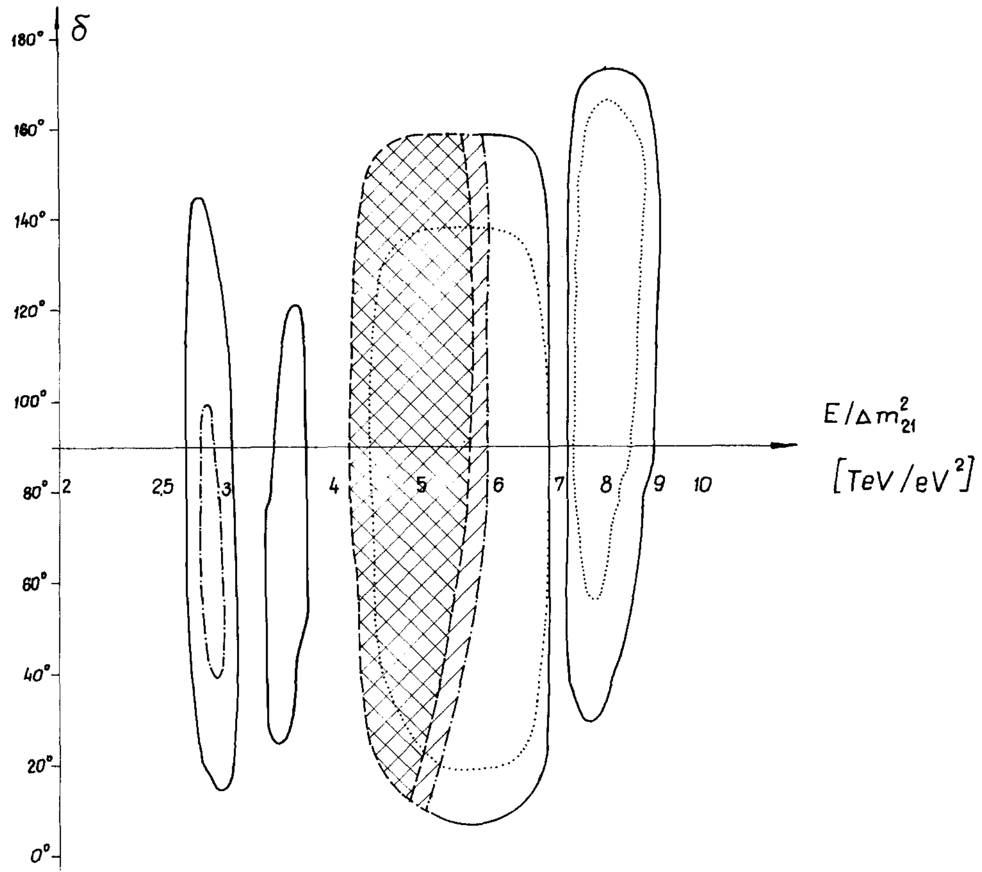

Fig. 3. Contours of constant $I_T=0.25$ (solid lines) and $I_T=0.50$ (dotted lines) as functions of δ and $E/\Delta m^2_{21}$ for $\sin\varphi_{12}=0.25$, $\sin\varphi_{13}=0.20$, $\sin\varphi_{23}=0.30$ and $\Delta m^2_{31}=3\Delta m^2_{21}$ ($|I_T|\geqslant 0.25$ and $|I_T|\geqslant 0.50$ for values of δ and $E/\Delta m^2_{21}$ lying inside the indicated contours). Shown are also the regions of values of δ and $E/\Delta m^2_{21}$ for which $\max(P(\nu_e\to\nu_\mu), P(\nu_\mu\to\nu_e))$ is not smaller than 0.15 (hatched area) and 0.20 (cross-hatched area).

$P(\nu_\mu\to\nu_e)$ on $E/\Delta m^2_{21}$ is shown for $\sin\varphi_{12}=0.25$, $\sin\varphi_{13}=0.2$, $\sin\varphi_{23}=0.3$, $\delta=\sqrt{1/3}$ and $\Delta m^2_{31}=3\Delta m^2_{21}$. As is seen from fig. 2, one can have $|I_T|\geqslant 0.2$, in relatively large intervals of $E/\Delta m^2_{21}$. In this example $4\mathcal{J}^V\cong 0.045$ and for values of $E/\Delta m^2_{21}$, e.g., in the range of ~ 2.5 TeV/eV^2 $|A^m_T|$ exceeds $|A^V_T|$ at least by a factor of 3.

Finally, the dependence of the asymmetry I_T on the vacuum phase δ is illustrated in fig. 3 for $\sin\varphi_{12}=0.25$, $\sin\varphi_{13}=0.20$, $\sin\varphi_{23}=0.30$ and $\Delta m^2_{31}=3\Delta m^2_{21}$. For values of δ and $E/\Delta m^2_{21}$ from the regions bounded by the solid (dotted) lines we have $|I_T|\geqslant 0.25$ ($|I_T|\geqslant 0.50$). The dashed and dash-dotted contours limit the regions of values of δ and $E/\Delta m^2_{21}$ wherein $\max(P(\nu_e\to\nu_\mu), P(\nu_\mu\to\nu_e))\geqslant 0.15$ and $\max(P(\nu_e\to\nu_\mu), P(\nu_\mu\to\nu_e))\geqslant 0.20$, respectively.

To summarize, our results indicate that there exist relatively large regions of values of the parameters characterizing the three-neutrino oscillations, for which the neutrino transitions and the T-violation effects in the three-neutrino oscillations can be considerably enhanced when neutrinos cross the earth. If the neutrino oscillations exist at the observable level, the T-violation effects in neutrino oscillations may turn out to be large enough to be detectable only in beams of neutrinos which have passed through the earth.

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Note added. While this work was in progress, we have received a paper by Kuo and Pantaleone [21] in which some aspects of the problem of *T*-violation effects in three-neutrino oscillations studied in the present work are also considered.

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