Calculating Generalised Gradients from Elliptical Field Maps

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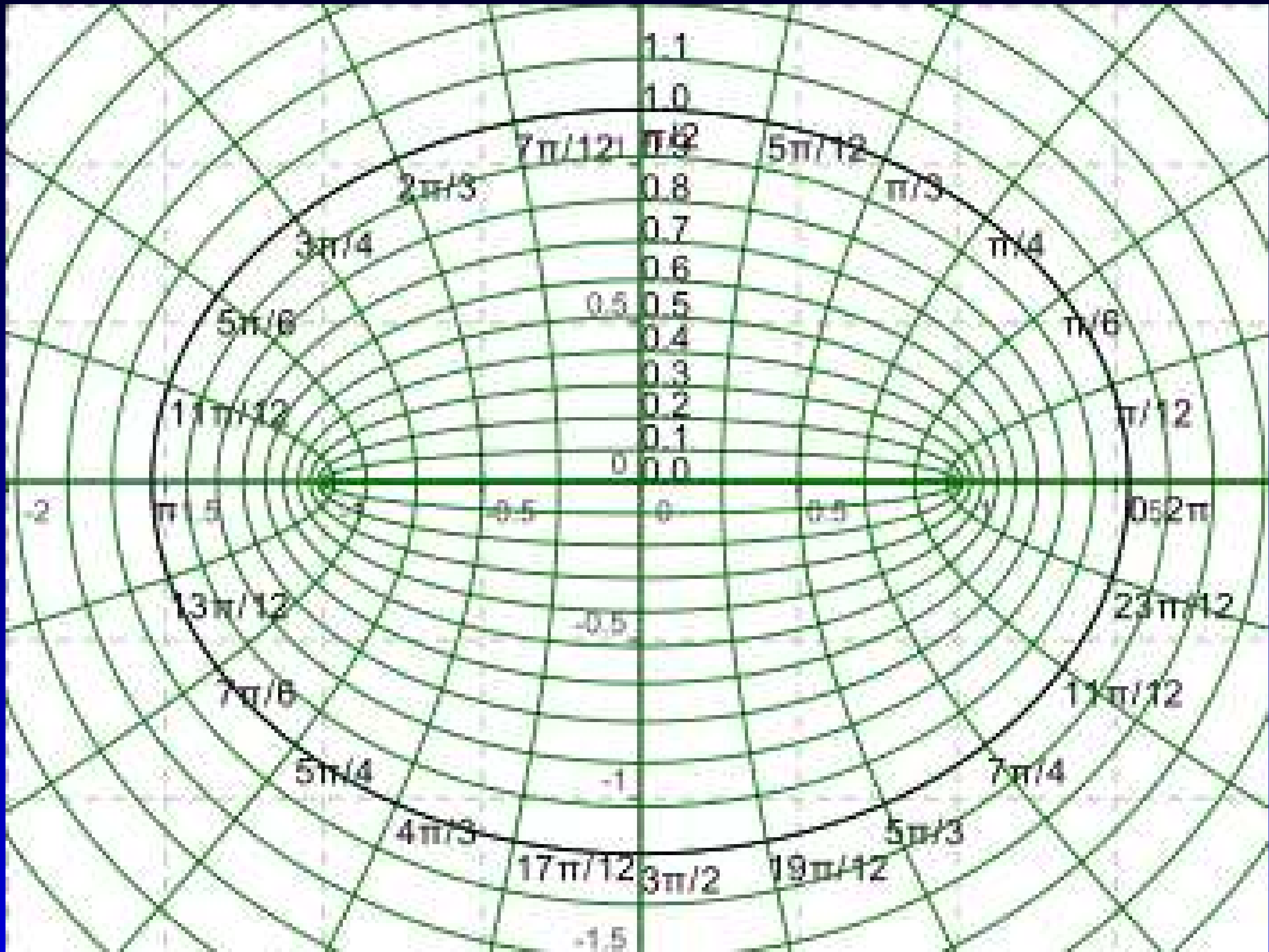
Previous Work

- Characterise an arbrtrary magnetic field in terms of it's multipole expansion and generalised gradients to produce an analytical description of field as a fuction of the longitudinal coordinate
- Use the analytical expression in differential algebra or Lie algebra code to generate a Taylor or Lie (symplectic) map for the dynamics in the magnet.
- Evaluate the analystical expressions to perform a numerical integration giving a fast particle tracking code to describe the evolution of the canonical coordinates within the magnet.
- The C++ code that has been has been written has a modular structure which facilitates extending the code
- A Synchrotron Radiation Module is being implemented which calculates the synchrotron emission from a particle into an arbitrary aperture
- eg ILC Helical undulator

Advantages of an Elliptical Field Map

- The accuracy of the analytical field, increases exponentially inside the initial cylinder.
  - It helps to have the initial cylinder as large as possible.
- In many situations an elliptical field map has advantages:
  - Wiggler sytems, where the gap height is much smaller then the horizontal aperture.
  - EMMA: The beam excursion is larger horizontally, than vertically.

Elliptical Coordinate System (u,v)



Ellipses have common foci at $\pm f$

The General Scalar Potential

$$x = f\ \cosh(u)\cos(v), \quad y = f\ \sinh(u)\sin(v)$$

In cylindrical coordinates:

$$\Psi(x,y,z) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk\, G_m(k)\ \exp(\imath kz)\exp(\imath m\phi) I_m(k\rho)$$

In elliptical coordinates:

$$\Psi(x,y,z) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk\, G_m(k)\ \exp(\imath kz) Ce_m(u,q) ce_m(v,q)$$

$G_m(k)$ are arbitrary coefficients, and the product $Cer(u,q)cer(v,q)$ forms a complete analytical function in (x,y) - similarly with $\exp(\imath m\phi)I_m(k\rho)$.
$Ce_m(u,q)$ and $ce_m(v,q)$ are Mathieu functions,
$q = -\frac{k^2 f^2}{4}$ is related to the longitudinal wave vector, k

Connection Coefficients

Additionally, there exists identities between elliptical and cylindrical functions:

$$Ce_r(u,q)ce_r(v,q) = \sum_{m=0}^{\infty} \alpha_m^r(k) I_m(k\rho) cos(m\phi)$$

and

$$Se_r(u,q)se_r(v,q) = \sum_{m=0}^{\infty} \beta_m^r(k) I_m(k\rho) sin(m\phi)$$

The key to calculating generalised gradients from an elliptical fieldmap lies in solving the Mathieu equations and calculating the connection coefficients, $\alpha_m^r(k)$ and $\beta_m^r(k)$.

Mathieu Functions

$$\frac{d^2Q}{du^2} + [a - 2q\cos(2v)]Q = 0$$

also, the modified Mathieu function:

$$\frac{d^2P}{du^2} - [a - 2q\cosh(2u)]P = 0$$

We need solutions to the Mathieu equation that are periodic in 2π and these solutions only exist for certain specific values of the separation constant, a.

There are two sets of solutions that are even or odd, $a_n(q)$ and $b_n(q)$

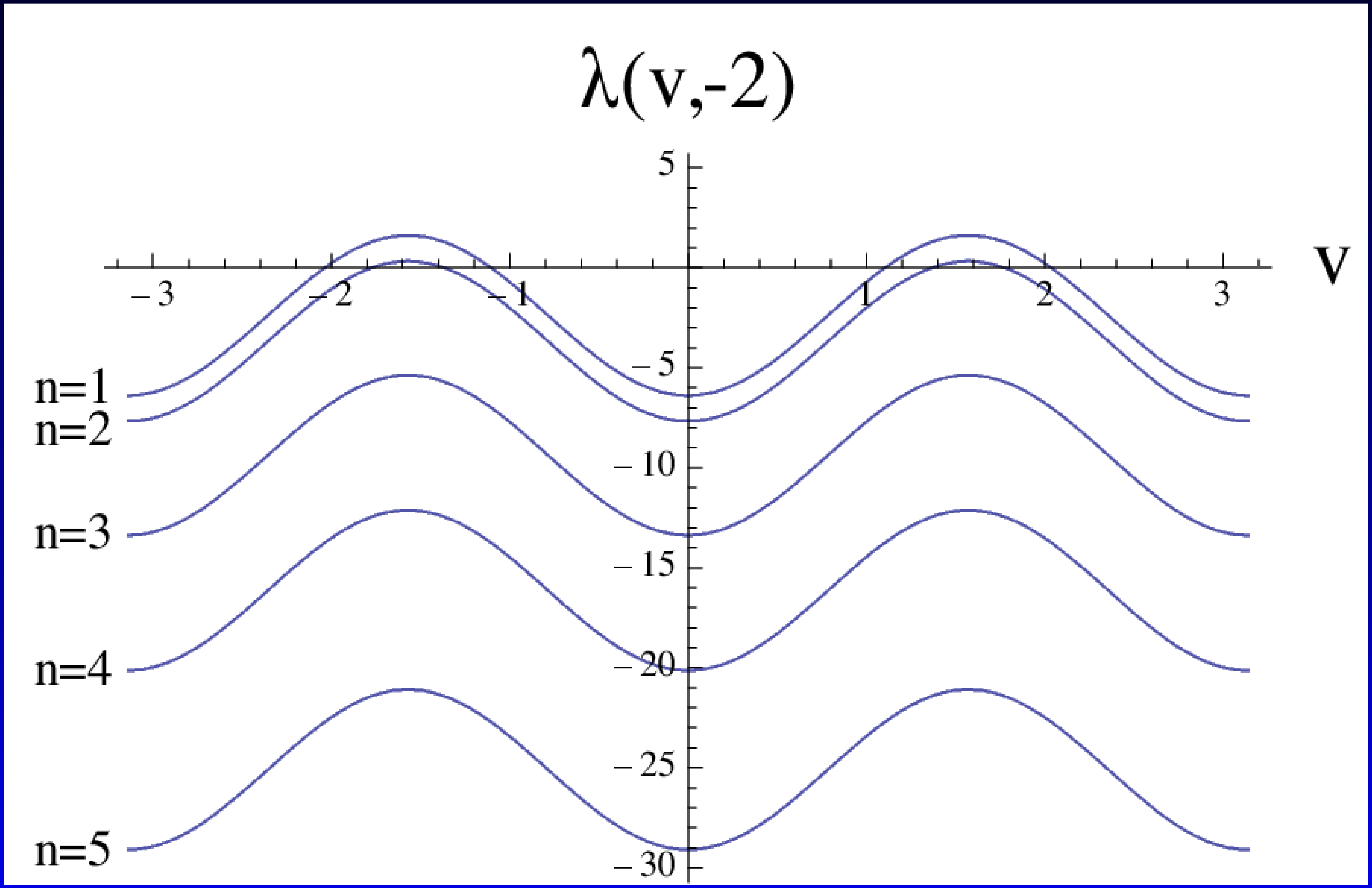
Note that if $\lambda = -[a - 2q\cos(2v)]$ the Mathieu equation can be written

$$\frac{d^2Q}{du^2} - \lambda Q = 0$$

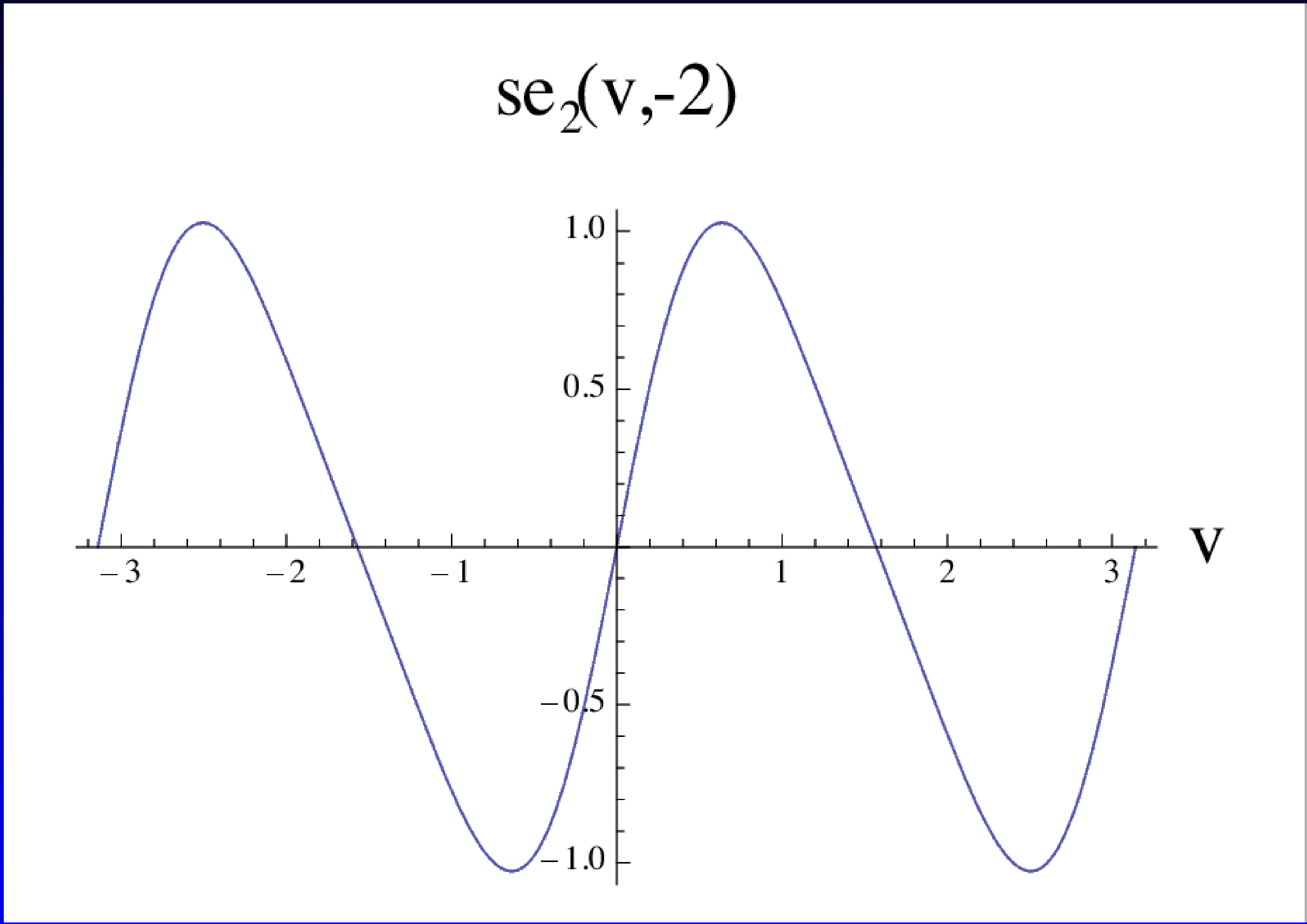
which if $\lambda < 0$ gives the equation for a harmonic oscillator, so we would expect oscillatory behaviour from the solution.

If $\lambda > 0$, the equation is similar to the Schrodinger equation in a tunnelling region, so we would expect the solution to decay exponentially.

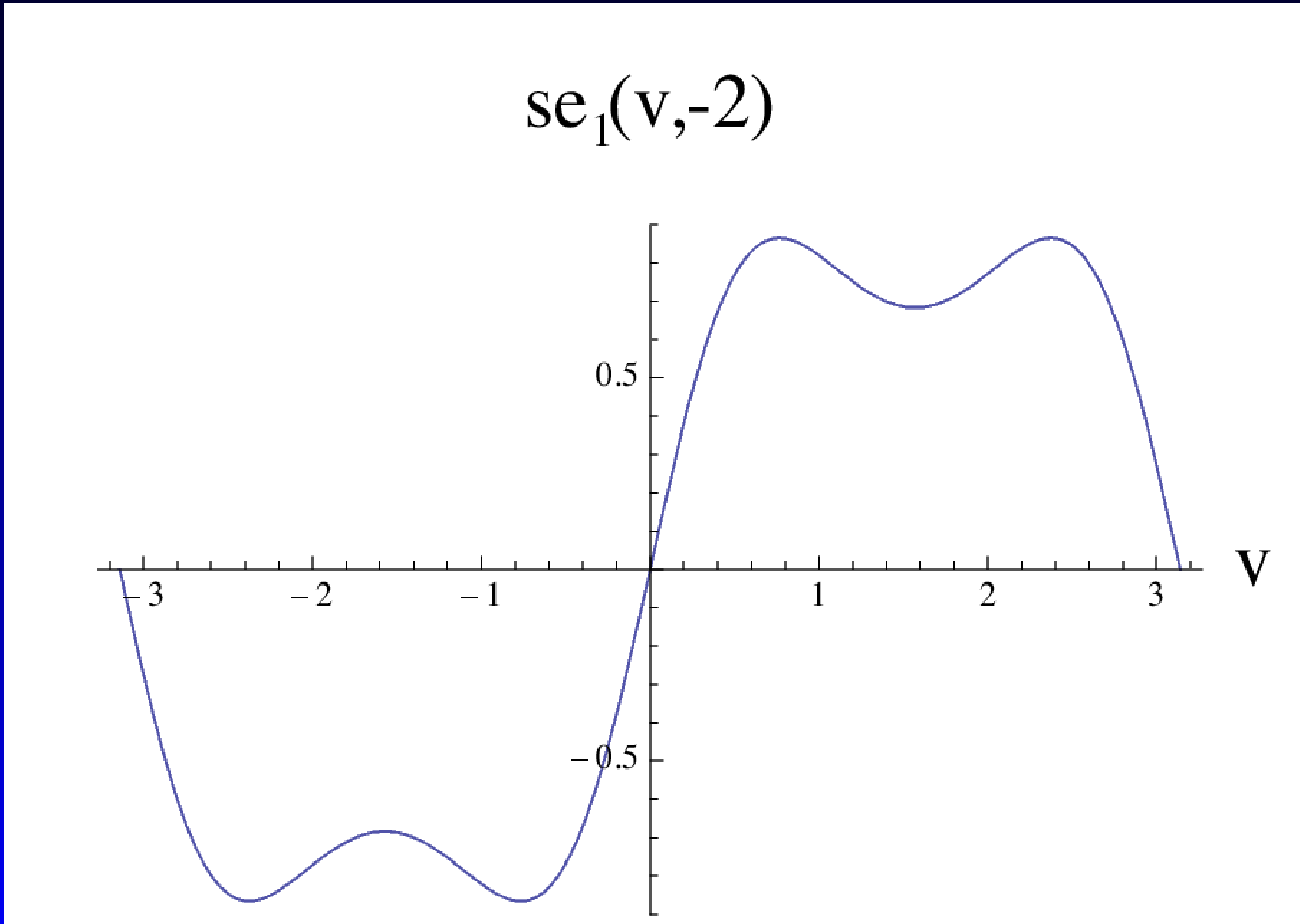
λ functions for $se_n(v, -2)$



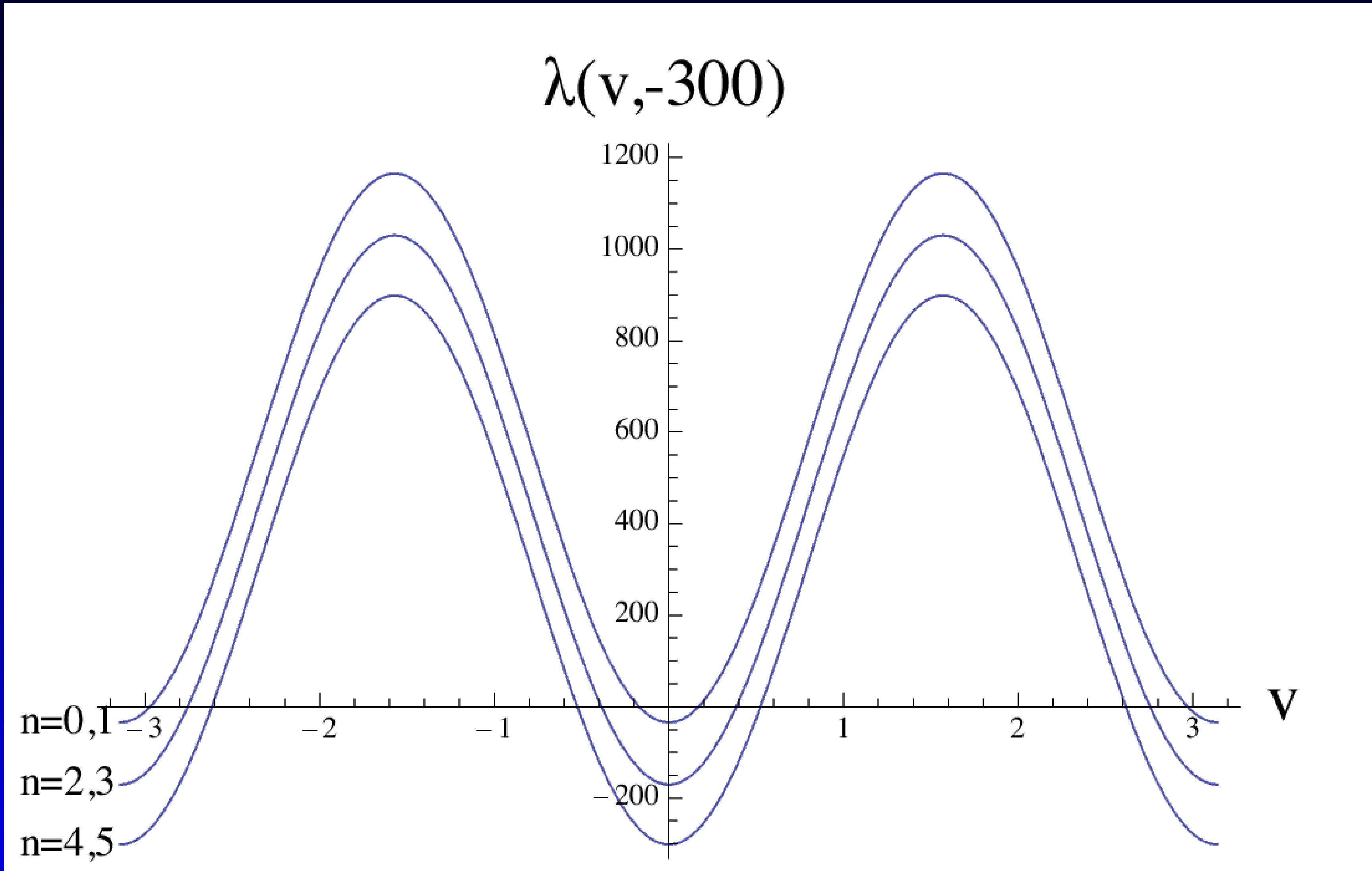
Mathieu Solutions, $se_2(v,-2)$



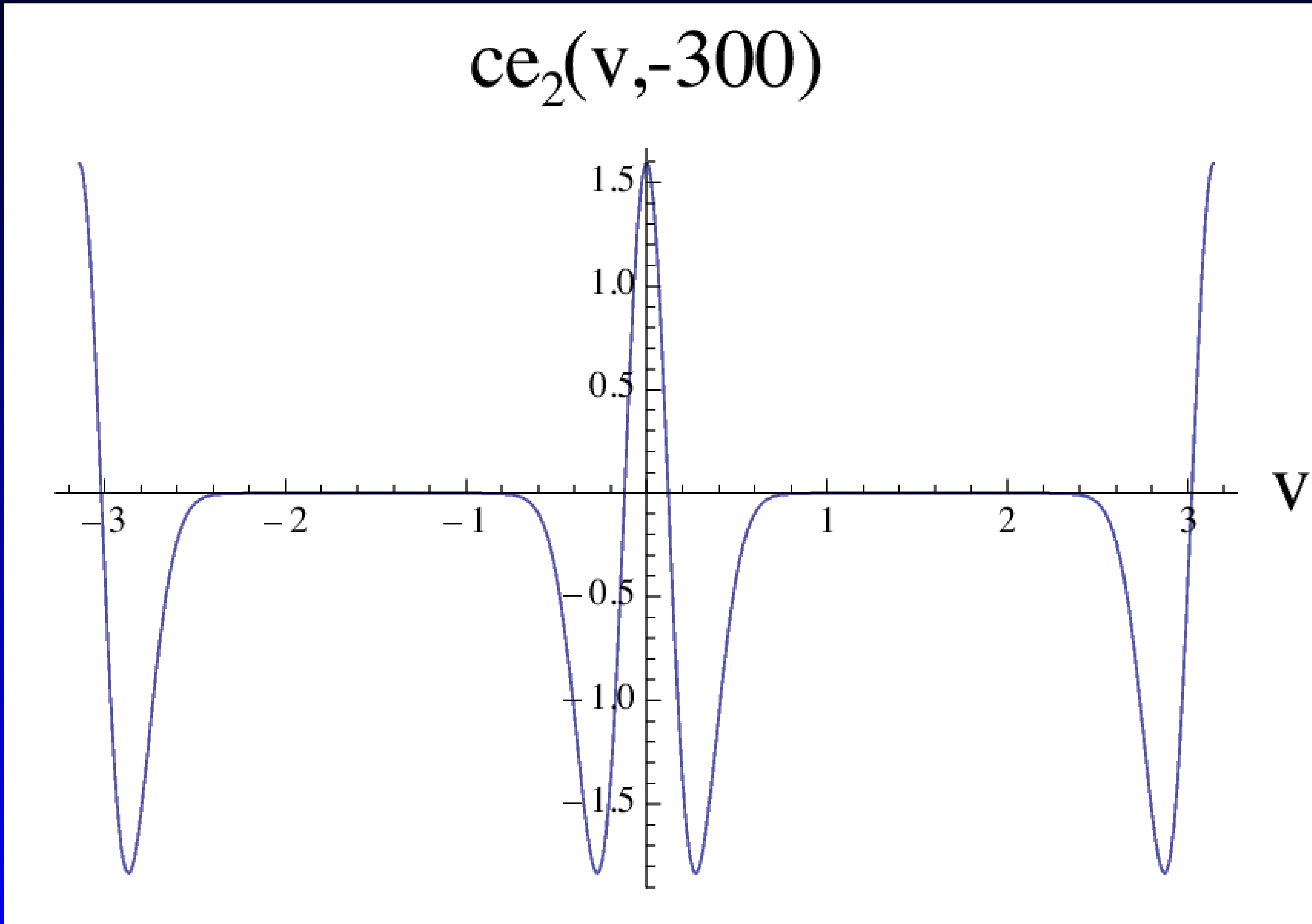
Mathieu Solutions, $se_1(v, -2)$



λ functions for $ce_n(v, -300)$



Mathieu Solutions, $ce_2(v, -300)$



Numerically Integrating Mathieu Equations

- An Adams predictor-corrector integrator was used to solve the Mathieu equations
  - Accurate to $\sim \mathcal{O}(h^{11})$
  - Probably overkill, but I had problems with lower order integrators - needs optimising
  - If you need a numerical integrator, see me
- By making use of symmetries in the solutions, we only need to integrate the Mathieu functions from 0 to $\pi/2$
- Furthermore, when the equation is in a 'tunneling' region we can insist that the function doesn't grow
- When the solution does integrate to 2π in a stable manner we can check the periodicity condition $Q(0, q) = Q(2\pi, q)$
  - Gives reassurance that the separation coefficients are calculated accurately

Normalising the Mathieu Equations

Like their trigonometric counterparts, the functions $ce_n(v,q)$ and $se_n(v,q)$ and are normalised so

$$\int_0^{2\pi} dv\, ce_m(v,q) se_n(v,q) \;=\; \pi\delta_{mn}$$

By simultaneaously solving this equation and the Mathieu equation, the normalisation constant can be found.
Similarly, the modified Mathieu equation needs integrating and normalising to find the solutions $Ce_m(u,q)$ and $Se_m(u,q)$.

Calculating the Connection Coefficients

There are well known equations for calculating the connection coefficients of the form:

$$\alpha^{2n+1}_{2m+1} = g^{2n+1}_c(k)\, A^{2n+1}_{2m+1}(q)$$

where,

$$g^{2n+1}_c(k) = [ce'_{2n+1}(\pi/2, q)ce_{2n+1}(0, q)]/[kf A^{2n+1}_1(q)]$$

and $A^n_m(q)$ is the Fourier coefficient of the Mathieu solution, ce_n, i.e.

$$ce_n(v, q) = \sum_{m=0}^{\infty} A^r_m(q) cos(mv)$$

Calculating the Generalised Gradients

Nearly ready to calculate the generalised gradients, C_m^l

In cylindrical form:

$$C_{m,s}^{[l]}(z) = \frac{i^l}{2^m m!} \int_{-\infty}^{\infty} dk \exp(ikz) k^{l+m-1} \frac{\hat{b}_m}{I'_m(kR)}$$

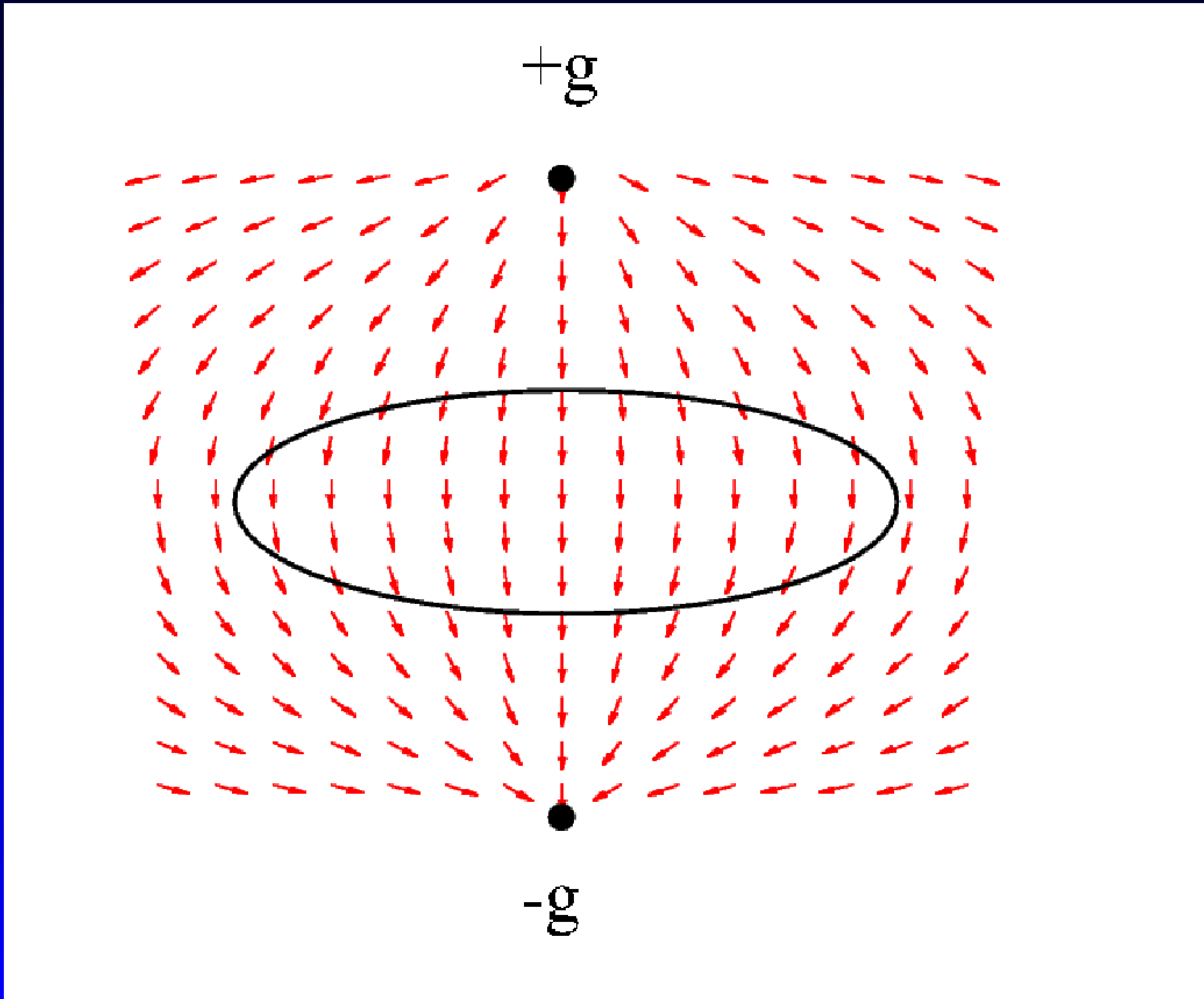
$\hat{b}_m$ are the 2D fourier coefficients of the field map. and in elliptical form:

$$C_{m,s}^{[l]}(z) = \frac{i^l}{2^m m!} \int_{-\infty}^{\infty} dk \exp(ikz) k^{l+m} \sum_{r=0}^{\infty} \beta_m^r(k) \frac{\mathcal{F}_r^s(k)}{Se'_r(U,q)}$$

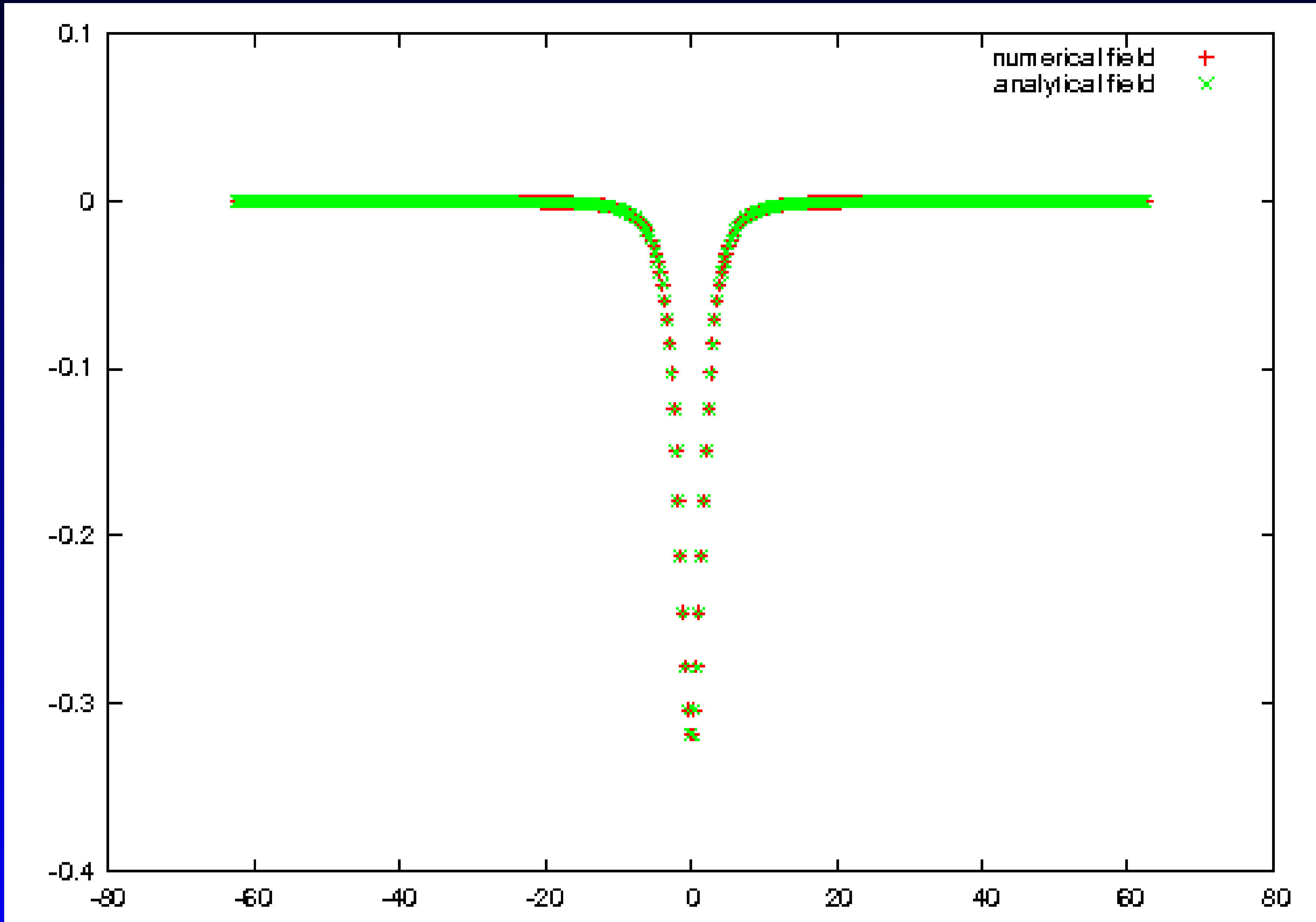
Here, to find $\mathcal{F}_r^s(k)$ we perform a Fourier transform in the longitudinal (z) axis, to find $\mathcal{F}(v,k)$ and in the elliptical (v) axis we perform the integration:

$$\mathcal{F}_r^s(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dv\, se_r(v,q) \mathcal{F}(v,k)$$

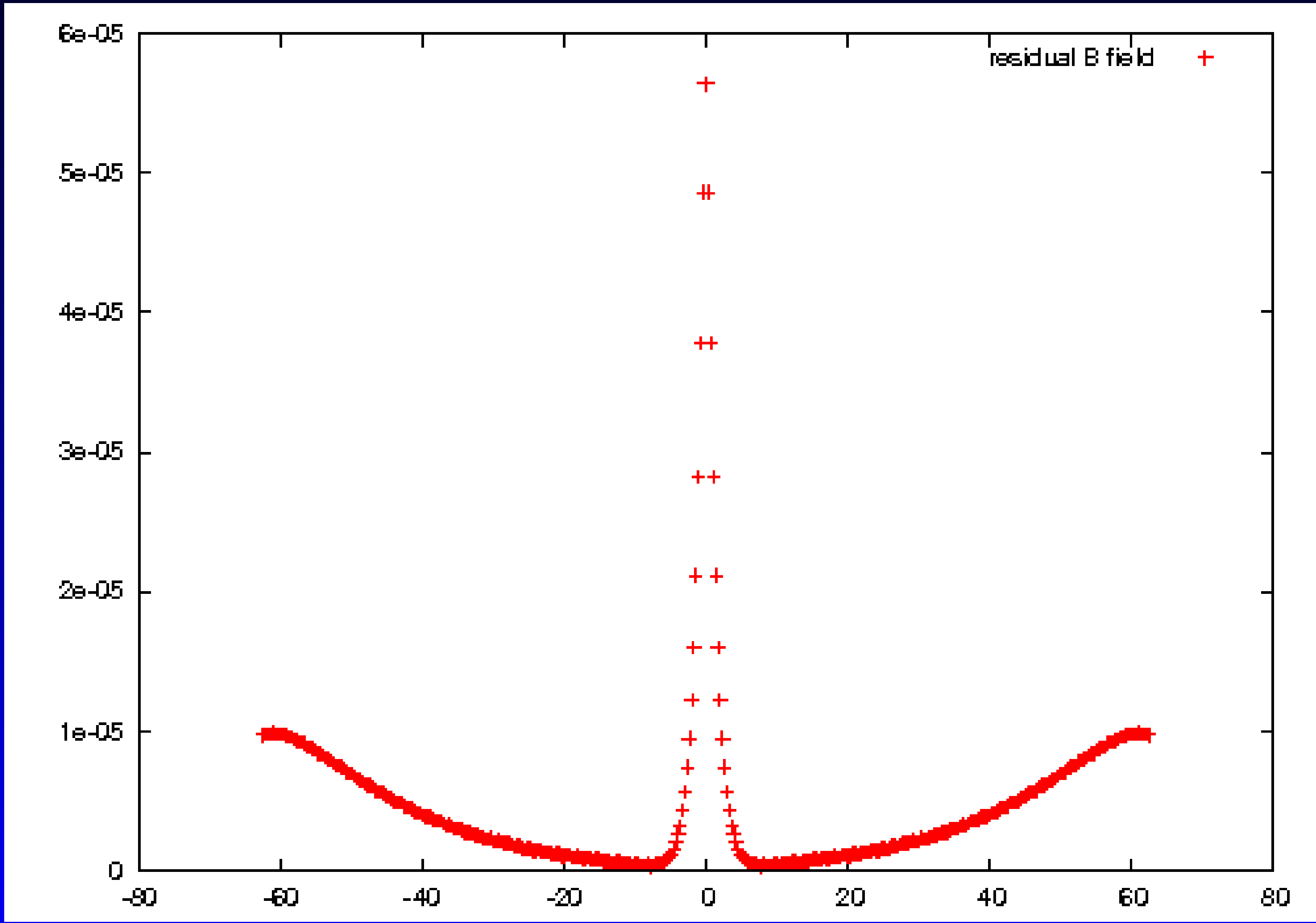
Numerical Benchmarkingwith a Monopole Doublet



On-Axis Field Comparison, B_y



On-Axis Field Comparison, B_y



Future Work

- Describe the Field from the EMMA magnets in terms of GGs
  - Presently, the large excursion transports the particle outside the bounding cylinder where numerical innacuracies grow unacceptably large
  - Comparison with Yoel's work
- Application for helical undulators?
- Work on synchrotron emission (ILC) undulators still ongoing