Numerical Techniques for Calculating Generalised Gradients

David Newton

Liverpool Group Meeting
18/02/2008

Generalised Gradients

For a periodic structure, a general scalar potential [a] that satisfies Laplace's equation, (a cylindrical harmonic or multipole expansion) can be written:

$$\Psi = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk\, G_m(k)\ \exp(\imath kz) \exp(\imath m\phi) I_m(k\rho)$$

I_m are the modified Bessel functions which can be expressed as a Taylor expansion:

$$I_m(x) = \sum_{L=0}^{\infty} \frac{1}{L!(m+L)!} \left(\frac{x}{2}\right)^{2L+m}$$

and $G_m(k)$ are arbitrary coefficients.
From this, the vector potentials can be written as:

$$A_\phi = 0$$

$$A_\rho = \sum_{m=1}^{\infty} \frac{\cos(m\phi)}{m} \rho \frac{\partial}{\partial z} \psi_{\omega,s} - \frac{\sin(m\phi)}{m} \rho \frac{\partial}{\partial z} \psi_{\omega,c}$$

$$A_z = \sum_{m=1}^{\infty} -\frac{\cos(m\phi)}{m} \rho \frac{\partial}{\partial \rho} \psi_{\omega,s} + \frac{\sin(m\phi)}{m} \rho \frac{\partial}{\partial \rho} \psi_{\omega,c}$$

[a] Alex J. Dragt - "Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics"

Numerical Techniques

- Cubic Spline Approximations
- Periodic Cubic Spline Approximation
- Fourier Transforms
    - Discrete and Spline Based
- Bessel Functions
- Numerical Benchmarking
    - Monopole Doublet Field

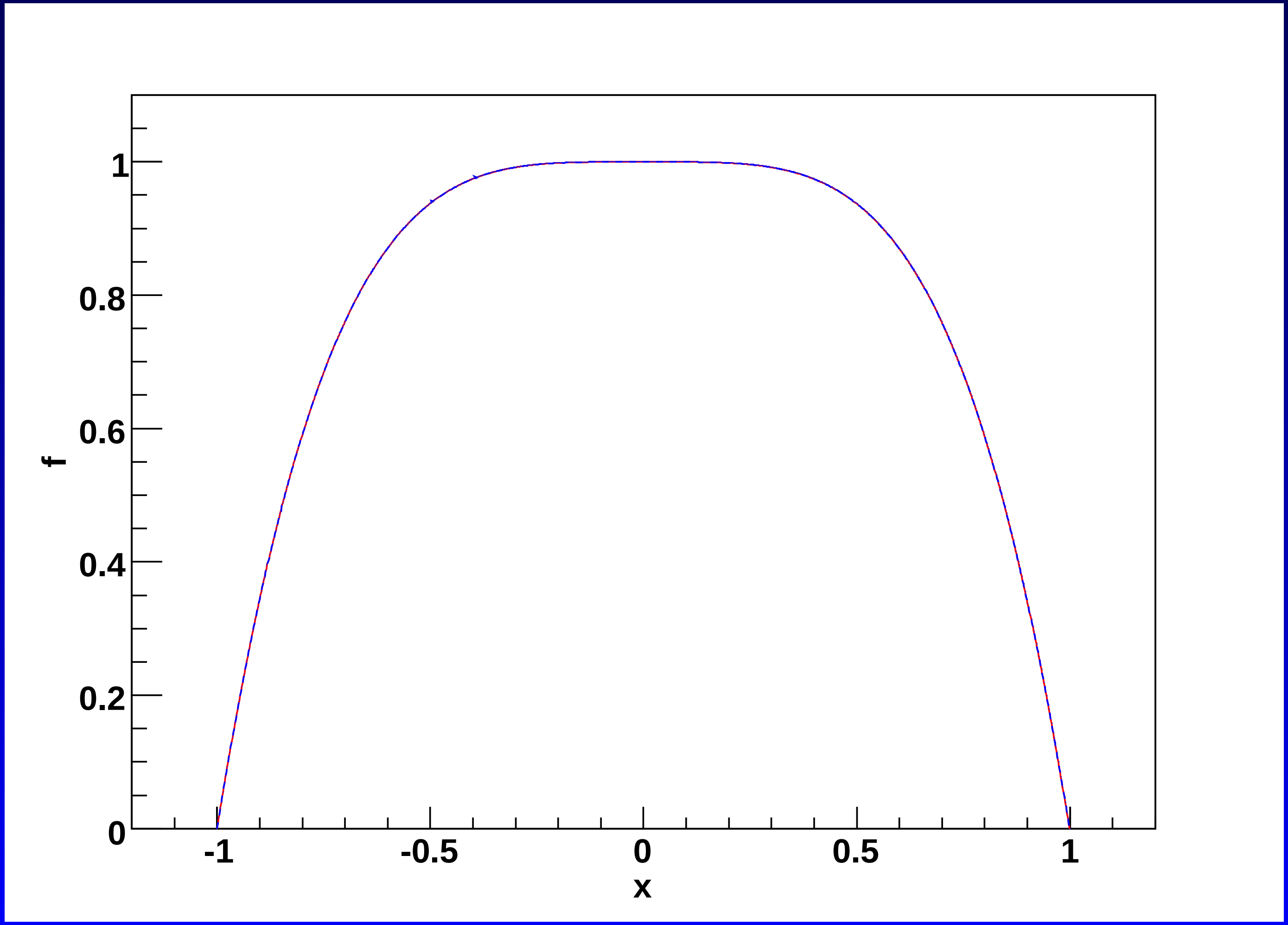
Cubic Spline Approximations

- Cubic splines are piecewise third-order polynomial fits, to a function $f(x)$, with specific continuity conditionsat each point x_i
- Polynomials on succesive intervals are matched such that f has continuous first and second derivatives at each point.
- The derivatives are calulated using the points either side of x_i
- This means at the end points the derivatives cannot be calculated
  - Source of numerical innaccuracy close to the endpoints

Test Function

$$f(x) = 1 - x^4$$

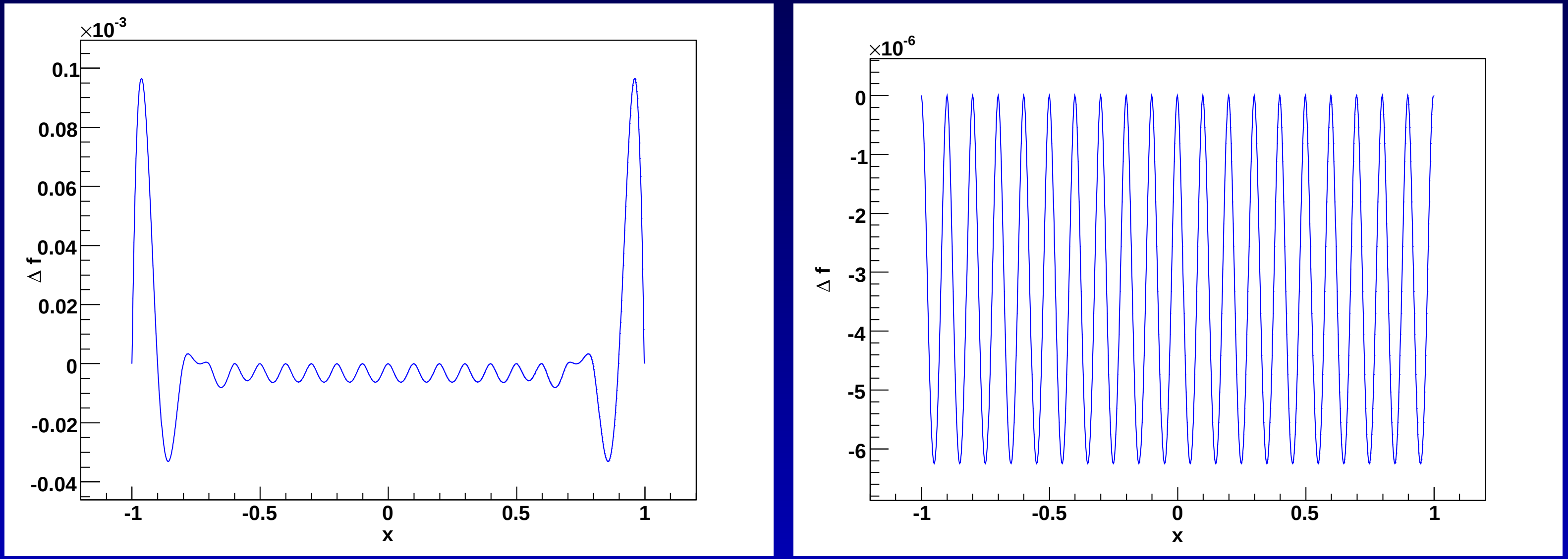
over the interval $[-1, 1]$



Test Function

$$f(x) = 1 - x^4$$

Residuals of the spline approximation



Large errors can occur at the end points

Periodic Spline Approximation

Fourier Transforms

Recall, that a general scalar potential can be written

$$\Psi = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk\, G_m(k)\, \exp(\imath k z) \exp(\imath m \phi) I_m(k\rho)$$

- This calls for an continuous fourier transform over the interval $[z = -\infty, z = \infty]$.
- If the field only has support over an interval [a,b], a definite integral can be calulated numerically
- A discrete fourier transform can be performed, however, this assumes the field is represented by a set of δ functions
- Using the spline approximation to the field, a continuous fourier transform can be evaluated exactly

Fourier Transforms

A cubic spline fits a polynomial, $g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ over each interval $[z_i, z_{i+1}]$. The fourier transform

$$\tilde{g}(k) = \frac{1}{2\pi} \int_{z_i}^{z_{i+1}} dz \, \exp(-\imath k z) g(z)$$

can be integrated by parts to give

$$\tilde{g}(k) = \frac{1}{2\pi} \frac{1}{\imath k} \exp(-\imath k z) g(z)|_{z_i}^{z_{i+1}} + \frac{1}{2\pi} \frac{1}{\imath k} \int_{z_i}^{z_{i+1}} dz \, \exp(-\imath k z) g'(z)$$

This expression contains two terms, the first part containing a power if $\frac{1}{\imath k}$ which can be solved exactly, and a second term involving an integra; over the first derivative $g'(z)$.

The second term can again be integrated by parts, to give a similar expression (with 1 more power of $\frac{1}{\imath k}$), and this process can be repeated as long as the higher derivatives of $g(z)$ exist.

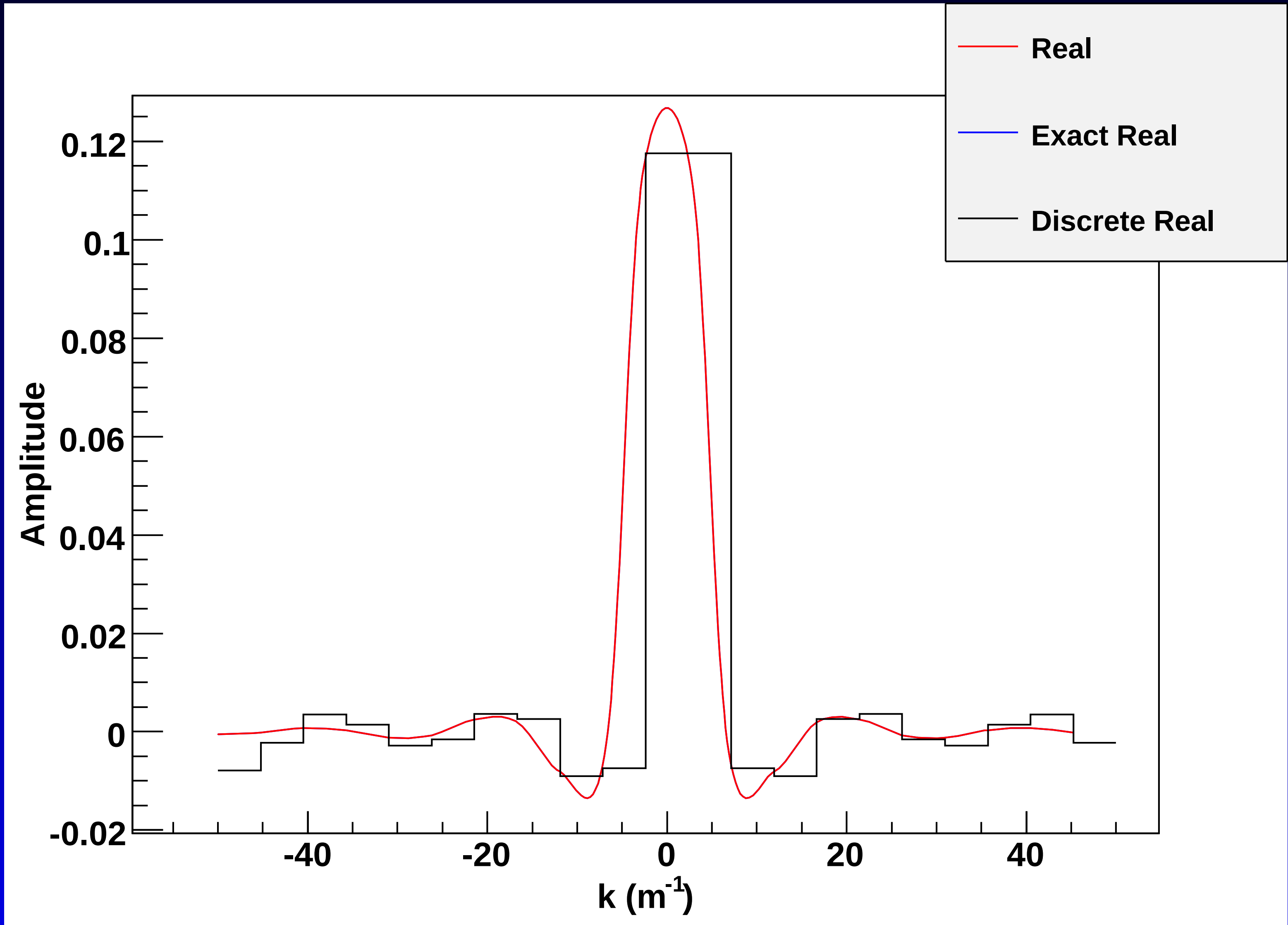
Fourier Transforms

For a cubic polynomial after 4 repetitions, the fourier transform of $g(z)$ calculated exactly.

$$\tilde{f}(k) = \frac{1}{2\pi} \int_a^{z_b} dz \, \exp(-\imath k z) f(z) = \sum_{n=0}^{N} \tilde{g}_n$$

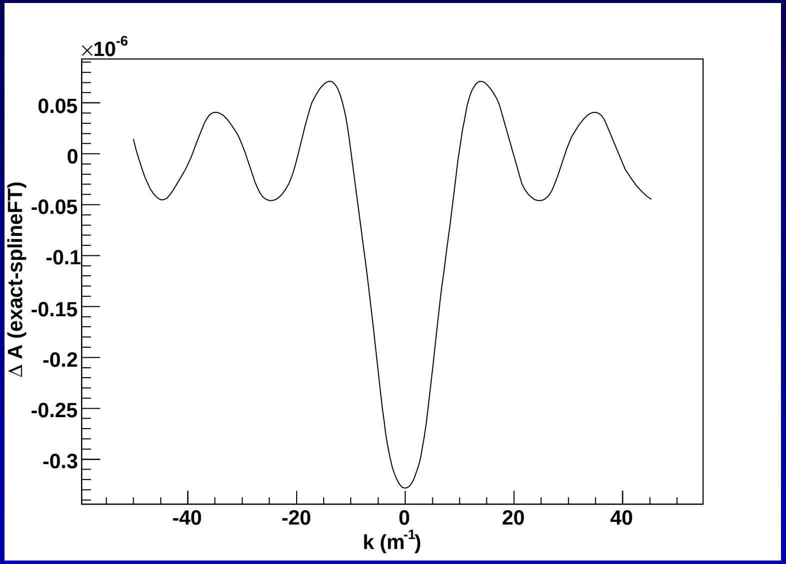
note that $\tilde{g}_n$ has four terms each contaning a further power of $\frac{1}{\imath k}$, which guarantees that $\tilde{f}(k)$ falls to zero for large k

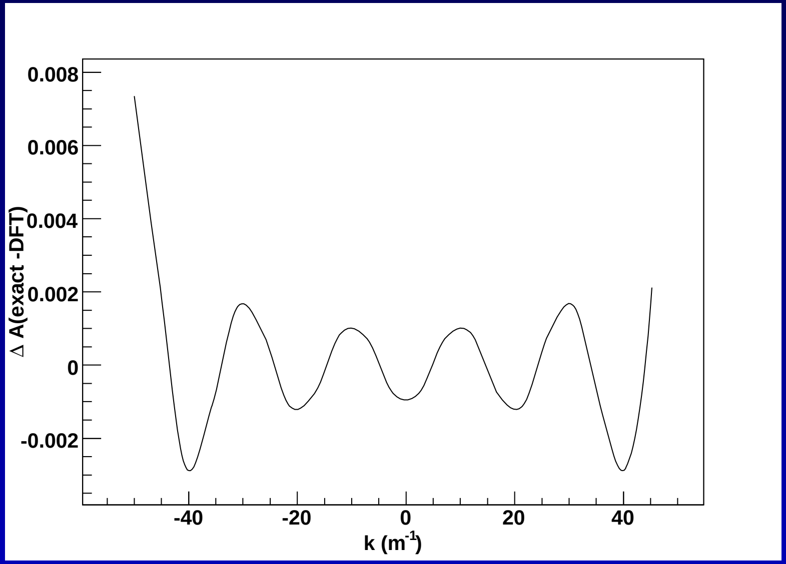
Fourier Transforms



Fourier Transforms

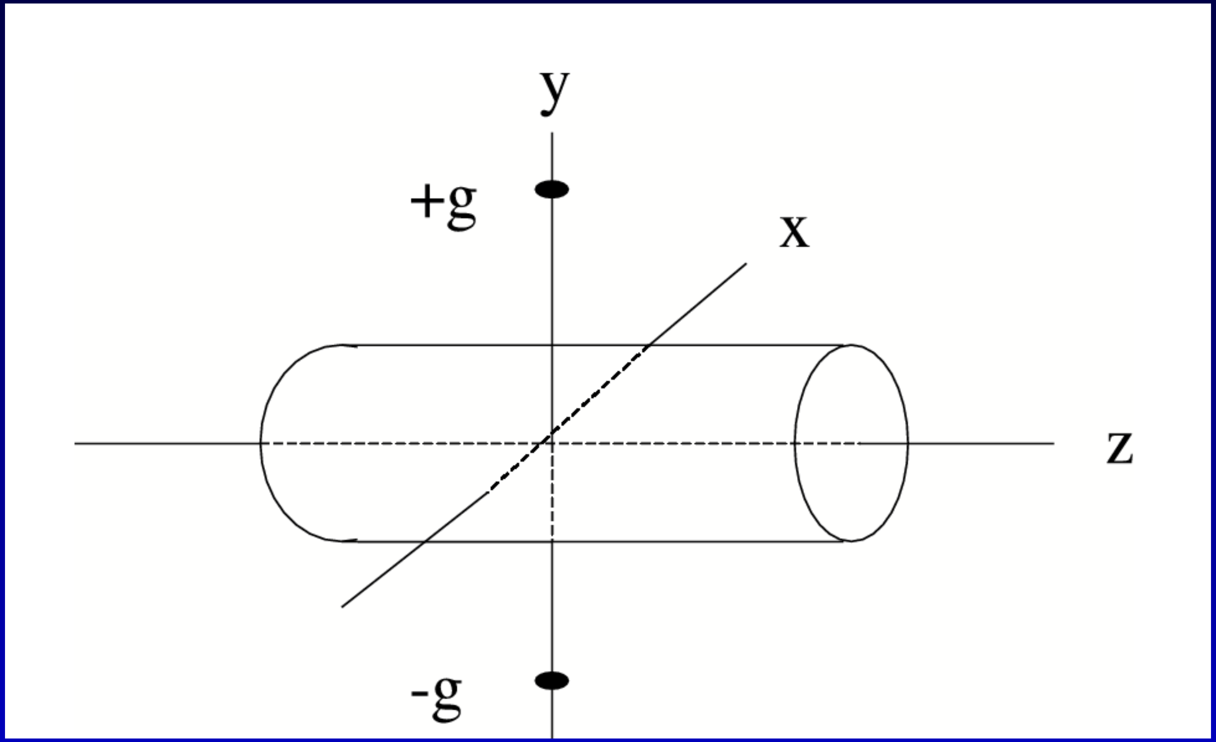
The spline fourier transform is four orders of magnitude more accurate than the DFT





Numerical Benchmarks

Consider a monopole doublet, with two magnetic monopoles of strength g:

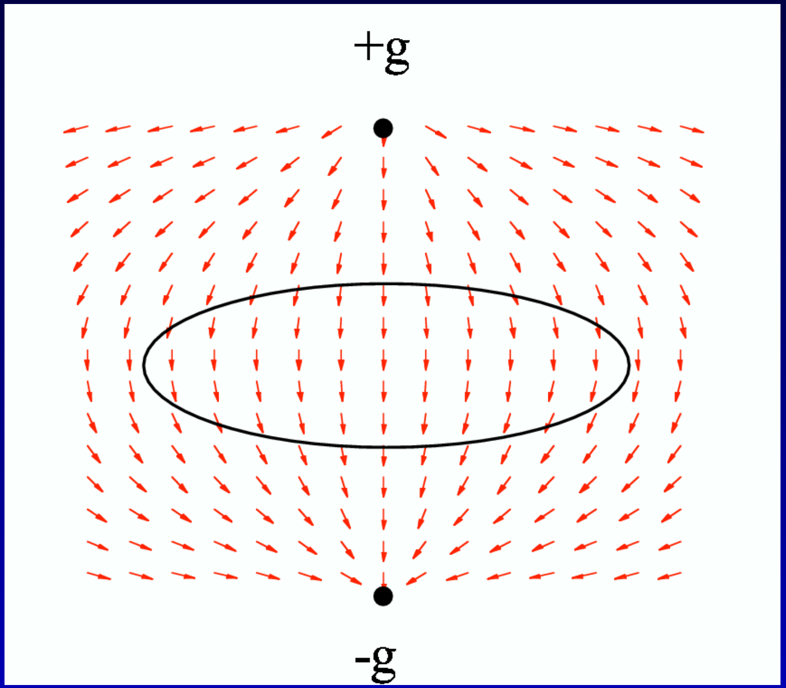


On the interior of a cylinder the field at all points satisfies

$\nabla \cdot B = 0$ and $\nabla \times B = 0$

Numerical Benchmarks

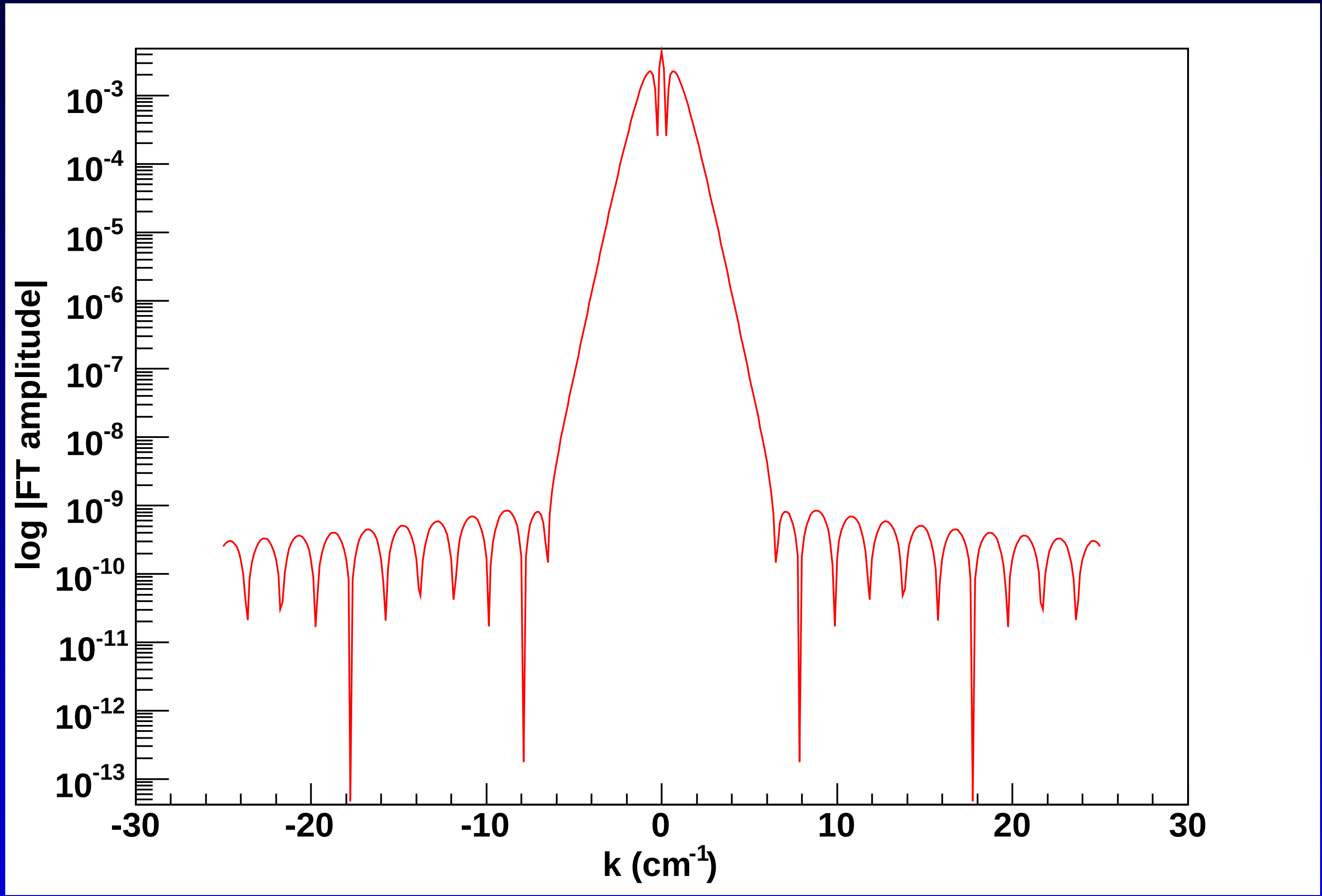
The field produced has a rapid spatial variation and is exactly solvable.



Furthermore, the fourier transform and the on-axis generalised gradients can be calculated analytically

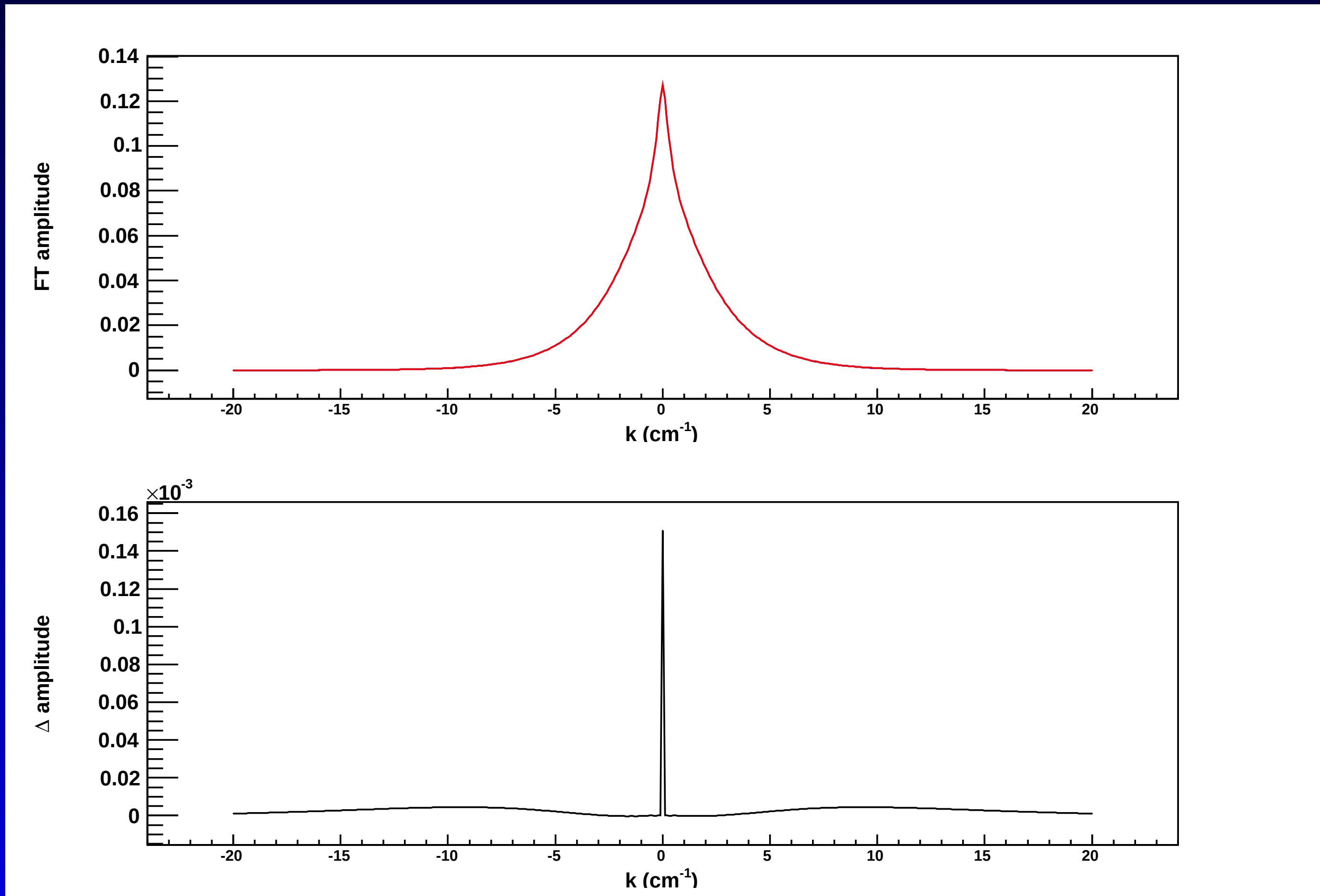
Monopole Doublet FT

m=1



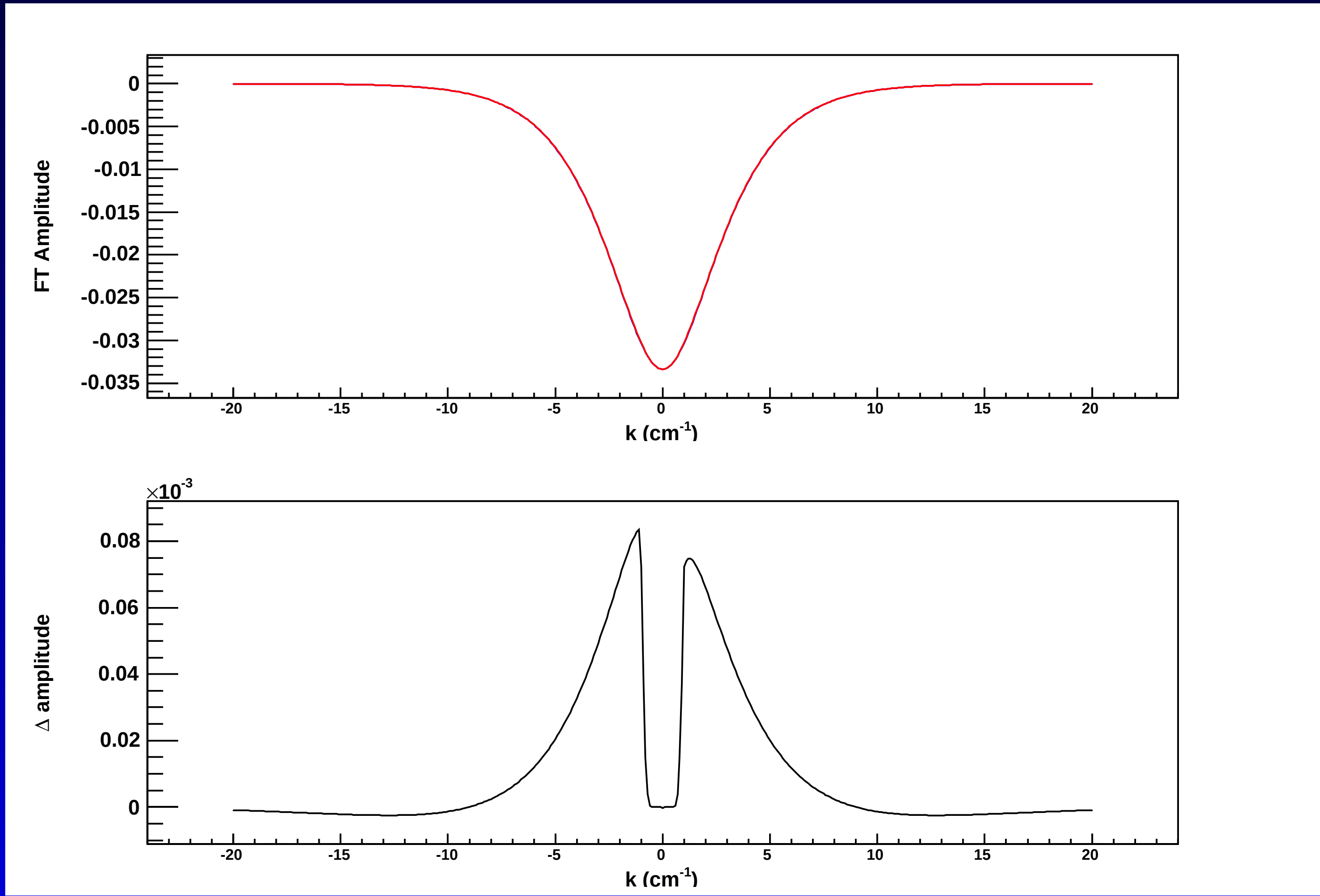
Monopole Doublet FT

m=1



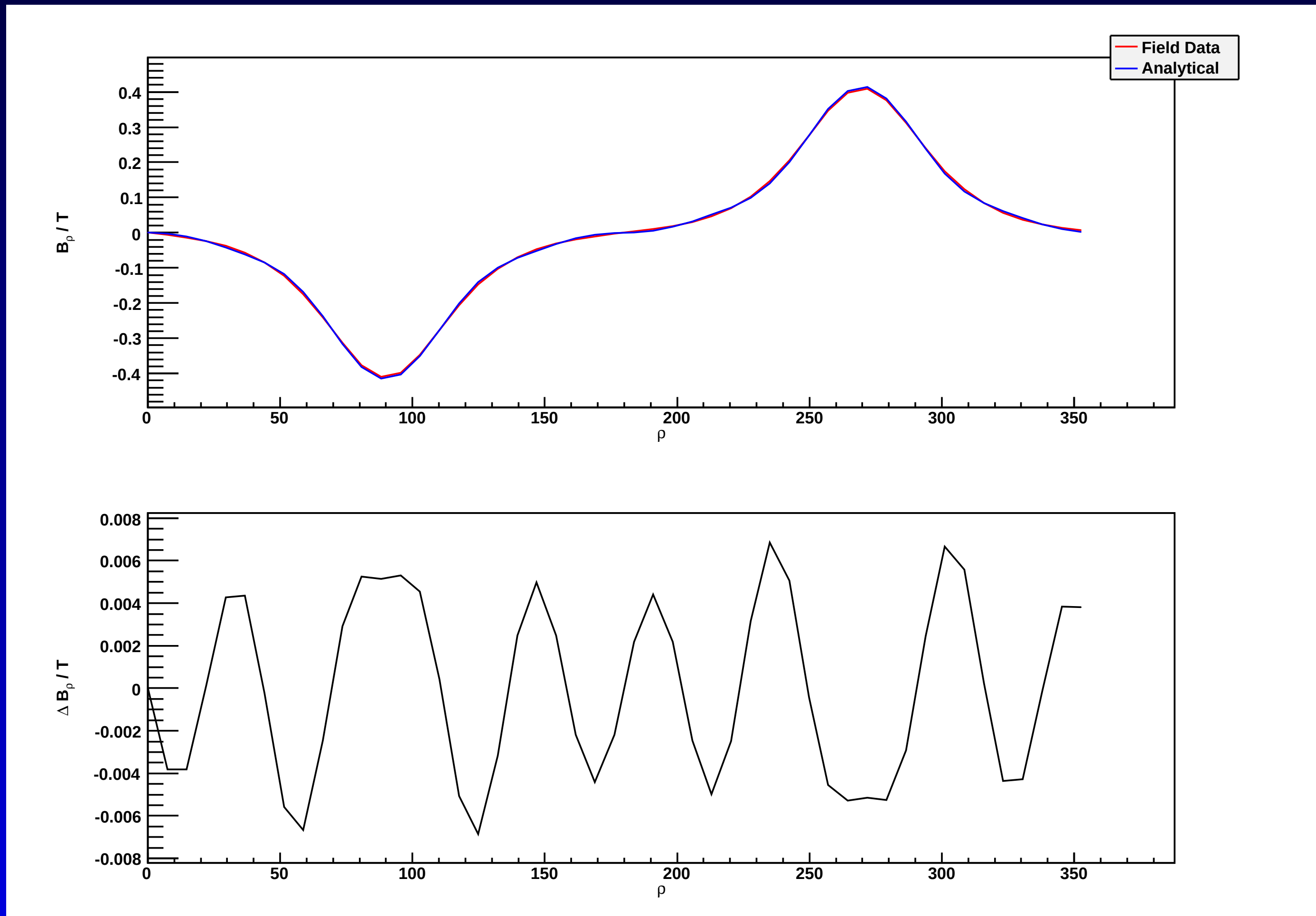
Monopole Doublet FT

m=7



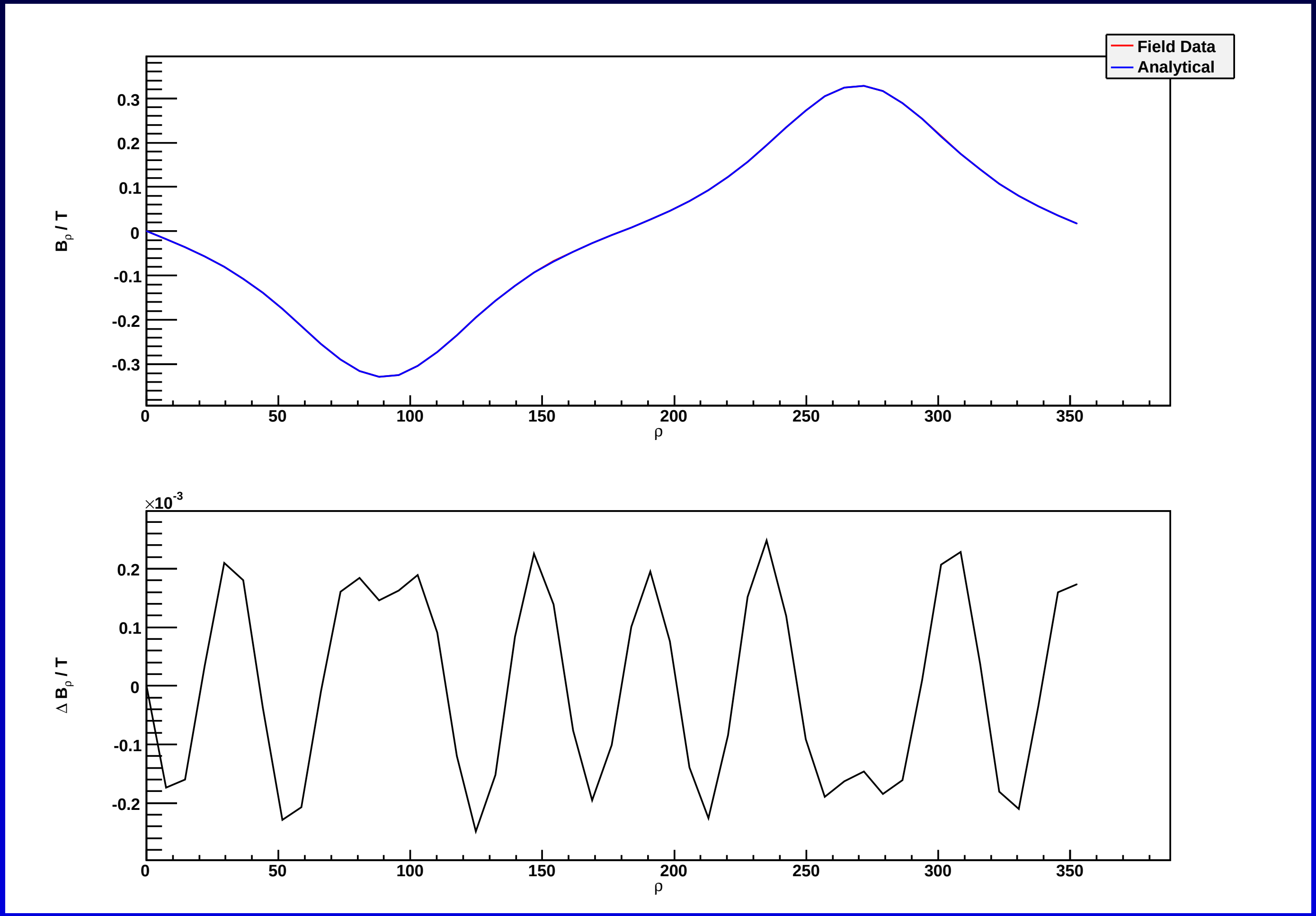
Monopole Doublet Field

$$\rho = 2.0$$



Monopole Doublet Field

$$\rho = 1.0$$



Monopole Doublet Field

$\rho = 0.5$