B Field Modelling and Generalised Gradients

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Generalised Gradients

For a periodic structure, of period λ_ω, a general scalar potential [a] can be written:

$$\Psi = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk \exp(ikz) I_m(k\rho)[(\hat{b}_m(k)\sin(m\phi) + \hat{a}_m(k) cos(m\phi)]$$

I_m are the modified Bessel functions which can be expressed as a Taylor expansion:

$$I_m(x) = \sum_{L=0}^{\infty} \frac{1}{L!(m+L)!} \left(\frac{x}{2}\right)^{2L+m}$$

and $\hat{a}_m$ and $\hat{b}_m$ are arbitrary coefficients.

From this, the vector potentials can be written as:

$$A_\phi = 0$$

$$A_\rho = \sum_{m=1}^{\infty} \frac{\cos(m\phi)}{m} \rho \frac{\partial}{\partial z} \psi_{\omega,s} - \frac{\sin(m\phi)}{m} \rho \frac{\partial}{\partial z} \psi_{\omega,c}$$

$$A_z = \sum_{m=1}^{\infty} -\frac{\cos(m\phi)}{m} \rho \frac{\partial}{\partial \rho} \psi_{\omega,s} + \frac{\sin(m\phi)}{m} \rho \frac{\partial}{\partial \rho} \psi_{\omega,c}$$

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[a] Alex J. Dragt - “Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics”

Field Mapping

Suppose the radial component of the magnetic field B_ρ is known, on the surface of a cylinder of radius R, then the field can be fitted in terms of a Fourier series:

$$B_\rho(\rho = R, \phi, z) = \sum_{m=0}^{\infty} \hat{b}_m(R, z)sin(m\phi) + \hat{a}_m(R, z)cos(m\phi)$$

The coefficients $\hat{a}_m$ ($\hat{b}_m$) correspond to normal (skew) components of the field, and the integer, m, is the order of the multipoles, i.e. m=0 corresponds to a solenoid component, m=1 represent the dipole component, m=2 a quadrupole etc.

Field Mapping

To calculate the generalised gradients, the Fourier series coefficients are scaled by the derivative of the Bessel function and a 'forward' fourier transform is performed.

$$C_{m,s}^{[L]}(z) = \frac{i^L}{2^m m!}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp(ikz)\frac{k^{L+m-1}}{I'_m(kR)}\,\hat{b}_m$$

and

$$C_{m,c}^{[L]}(z) = \frac{i^L}{2^m m!}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp(ikz)\frac{k^{L+m-1}}{I'_m(kR)}\,\hat{a}_m$$

$C_{m,\alpha}^{[L]}(z)$ is the L th derivative with respect to z of the generalised gradients $C_{m,\alpha}(z)$

Field Mapping

The generalised gradients can then be used to calculate the vector potential at any point within the volume of the cylinder, and thus the magnetic field, i.e.

$$B_\rho = \sum_{m=0}^{\infty} \sum_{L=0}^{\infty} (-1)^L \frac{m!(2L+m)}{2^{2L} L!(L+m!)}$$

$$\times C_{m,\alpha}^{[2L]} \rho^{2L+m-1} (\sin(m\phi) + \cos(m\phi))$$

note that along the z axis ($\rho = 0.0$), only the $m = 1, L = 0$ components contribute to the field, and the generalised gradients $C^0_{1,s}(z)$ and $C^0_{1,c}(z)$ correspond to the field components B_y and B_x along the z-axis.

Coordinate Transformation

Generally more usual to work in a Cartesian basis, so we need to transform the representation of the field from cylindrical coordinates (ρ, ϕ, z) to Cartesian (x, y, z).
The Generalised Gradients, give the on-axis field ($\rho = x = y = 0$) and are dependent only on z, so these do not need transforming.
The transformation can be achieved using the identities:

$$x = \rho \cos\phi \qquad y = \rho \sin\phi$$

$$\rho^{2L} = (x^2 + y^2)^L$$

$$\rho^m \cos m\phi = \Re(x + \imath y)^m \qquad \rho^m \sin m\phi = \Im(x + \imath y)^m$$

and

$$A_x = A_\rho \cos\phi, A_y = A_\rho \sin\phi$$

Coordinate Transformation

e.g for a pure, normal dipole (m=1), the scalar potential is:

$$\Psi(x, y, z) = \sin(\phi)\Psi_{1,s}(\rho, z)$$

where

$$\Psi_{1,s}(\rho, z) = C_{1,s}^{[0]}(z)\rho - (1/8)C_{1,s}^{[2]}(z)\rho^3 + (1/192)C_{1,s}^{[4]}(z)\rho^5 + \dots$$

$$\Psi(x, y, z) = yC_{1,s}^{[0]}(z) - (1/8)y(x^2+y^2)C_{1,s}^{[2]}(z) + (1/192)y(x^2+y^2)^2C_{1,s}^{[4]}(z) + \dots$$

These transformations give the scalar and vector potential as well as the magnetic field in Cartesian coordinates.

Guage Transformation

Finally, to simplify the integration process, a guage transformation can be made on the vector potential such that:

$$\mathbf{A}'(\mathbf{x},\mathbf{y},\mathbf{z}) + \nabla\lambda = \mathbf{A}(\mathbf{x}=\mathbf{0},\mathbf{y},\mathbf{z})$$

where

$$\lambda = \sum_{m=0}^{\infty} \Lambda_{m,c}(\rho,z)\cos m\phi + \sum_{m=1}^{\infty} \Lambda_{m,s}(\rho,z)\sin m\phi$$

and

$$\Lambda_{m,\alpha} = \sum_{L=0}^{\infty}(-1)^L \frac{m!}{2^{2L}L!(L+m)!} S_{m,\alpha}^{[2L]}(z)\rho^{(2L+m)}$$

Guage Transformation

eg, for a skew dipole

$$A'_x = xy(1/2)C^{[1]}_{1,c}(z) - (x^3y - xy^3)(1/24)C^{[3]}_{1,c}(z) + \dots$$

$$\frac{\partial\lambda}{\partial x} = 3xyS^{[0]}_{3,c}(z) - (1/16)(4x^3y + 12xy^3)S^{[2]}_{3,c}(z) + \dots$$

By setting $S^{[\alpha]}_{m+2,c}(z) = (1/12)C^{[\alpha+1]}_{m,s}(z)$ and summing $A'_x + \frac{\partial\lambda}{\partial x}$ the leading term of A_x disappears. i.e.

$$A_x = A'_x + \frac{\partial\lambda}{\partial x} = -(1/48)(x^3y - xy^3)C^{[3]}_{1,c}(z) + \dots$$

In the new guage $A'_y + \frac{\partial\lambda}{\partial y} = A_y$ and $A'_z + \frac{\partial\lambda}{\partial z} = A_y$.

This process can be iterated such that $A_x = 0$, to any arbitrary order

Symplectic Integrator

$$\mathcal{M} = \exp\left(: -\frac{\Delta\sigma}{2} P_z :\right) \exp\left(: -\frac{\Delta\sigma}{2} a_z :\right) \exp\left(: -\frac{\Delta\sigma}{2}\left(-\delta + \frac{P_x^2}{2(1+\delta)}\right) :\right)$$

$$A_y \exp\left(: -\Delta\sigma \frac{P_y^2}{2(1+\delta)} :\right) A_y^{-1} \exp\left(: -\frac{\Delta\sigma}{2}\left(-\delta + \frac{P_x^2}{2(1+\delta)}\right) :\right)$$

$$\exp\left(: -\frac{\Delta\sigma}{2} a_z :\right) \exp\left(: -\frac{\Delta\sigma}{2} P_z :\right)$$

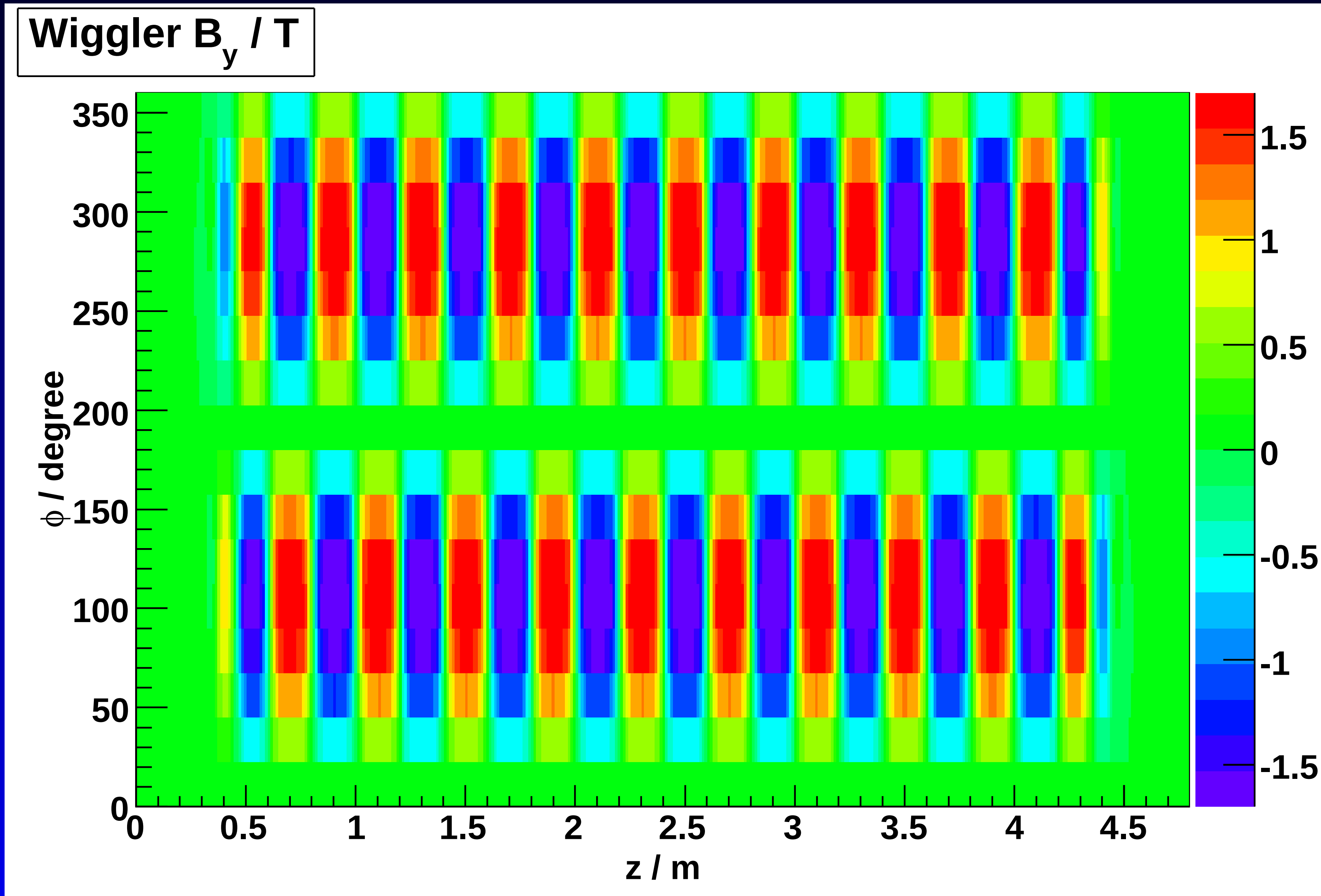
where $(- : H :)f = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$
and the operators A_y and A_y^{-1} involve the vector potential. [a]

Giving the transfer map:

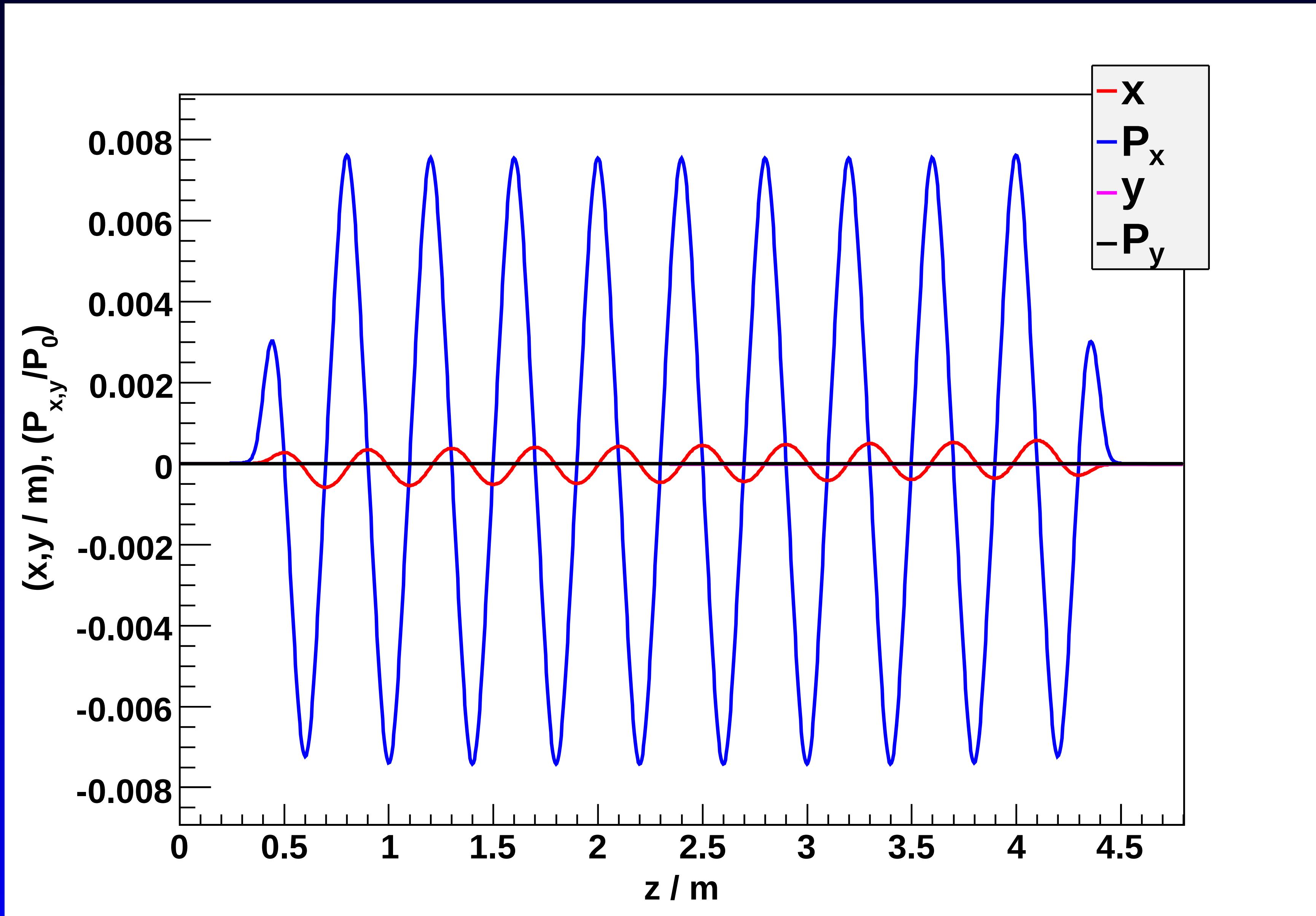
$$\mathcal{M} f_i(x, p_x, y, p_y, z, \delta) = f_f(x, p_x, y, p_y, z, \delta)$$

[a] Wu, Forest and Robin,2001

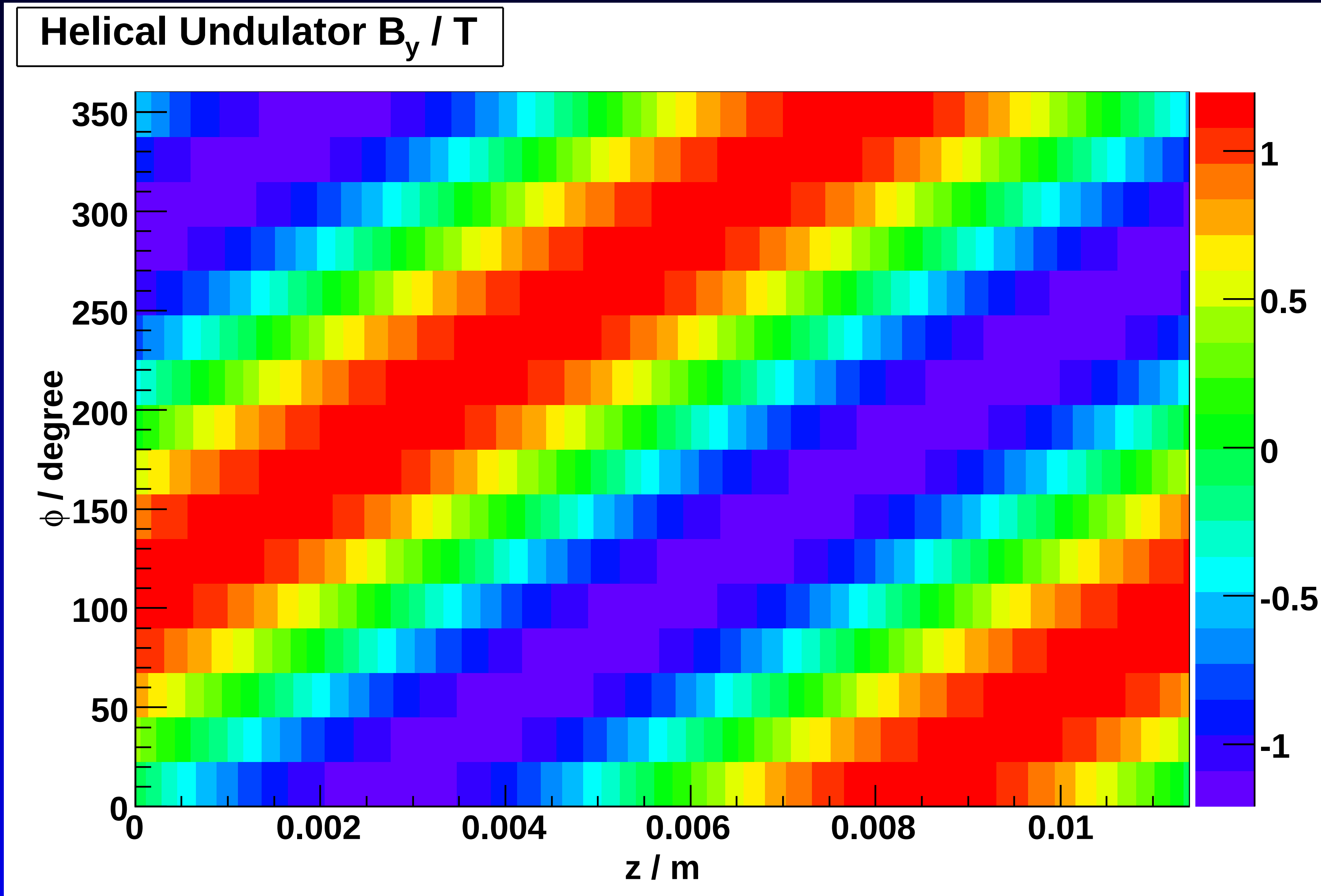
Cesr Wiggler Field Map



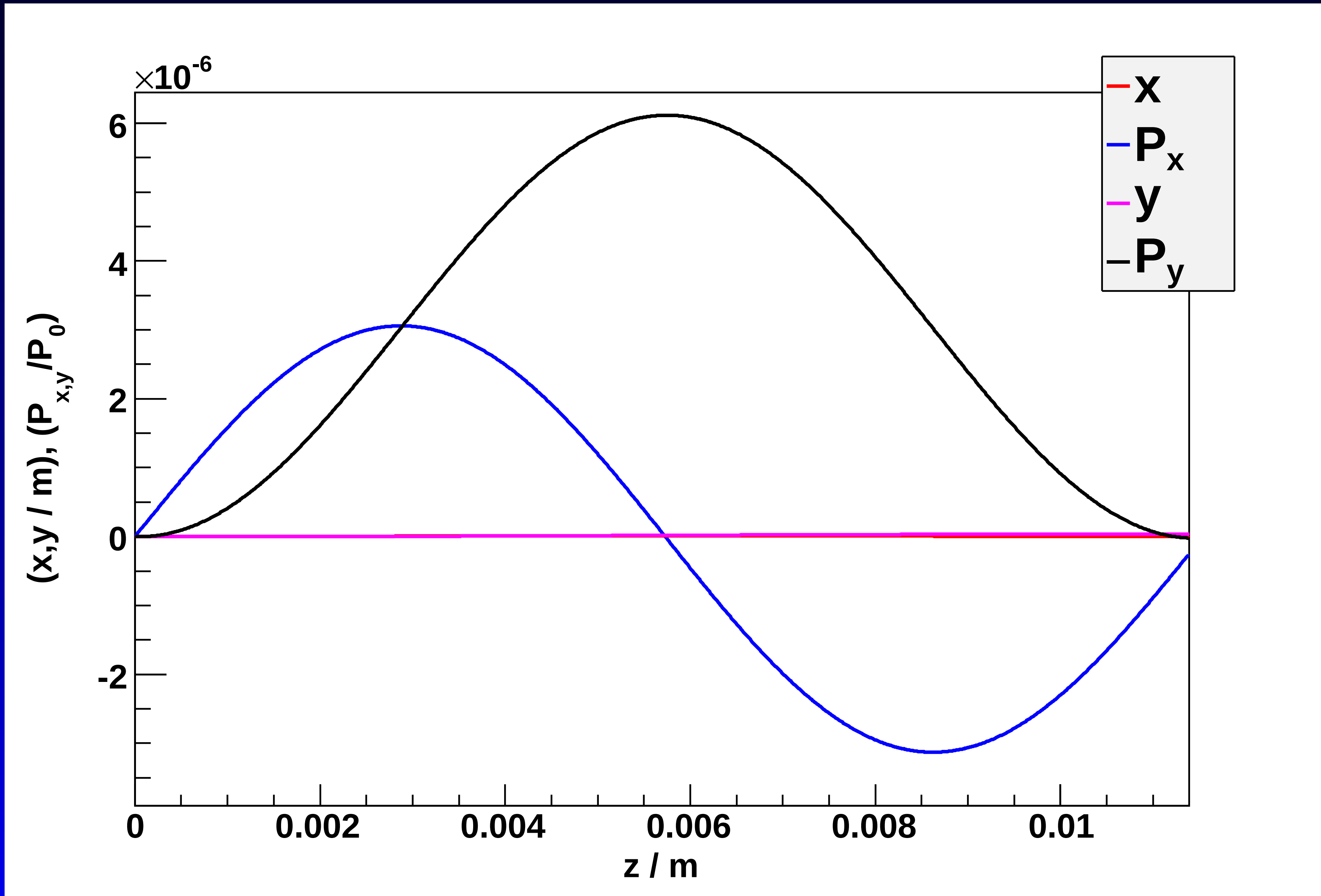
Cesr Wiggler Tracking



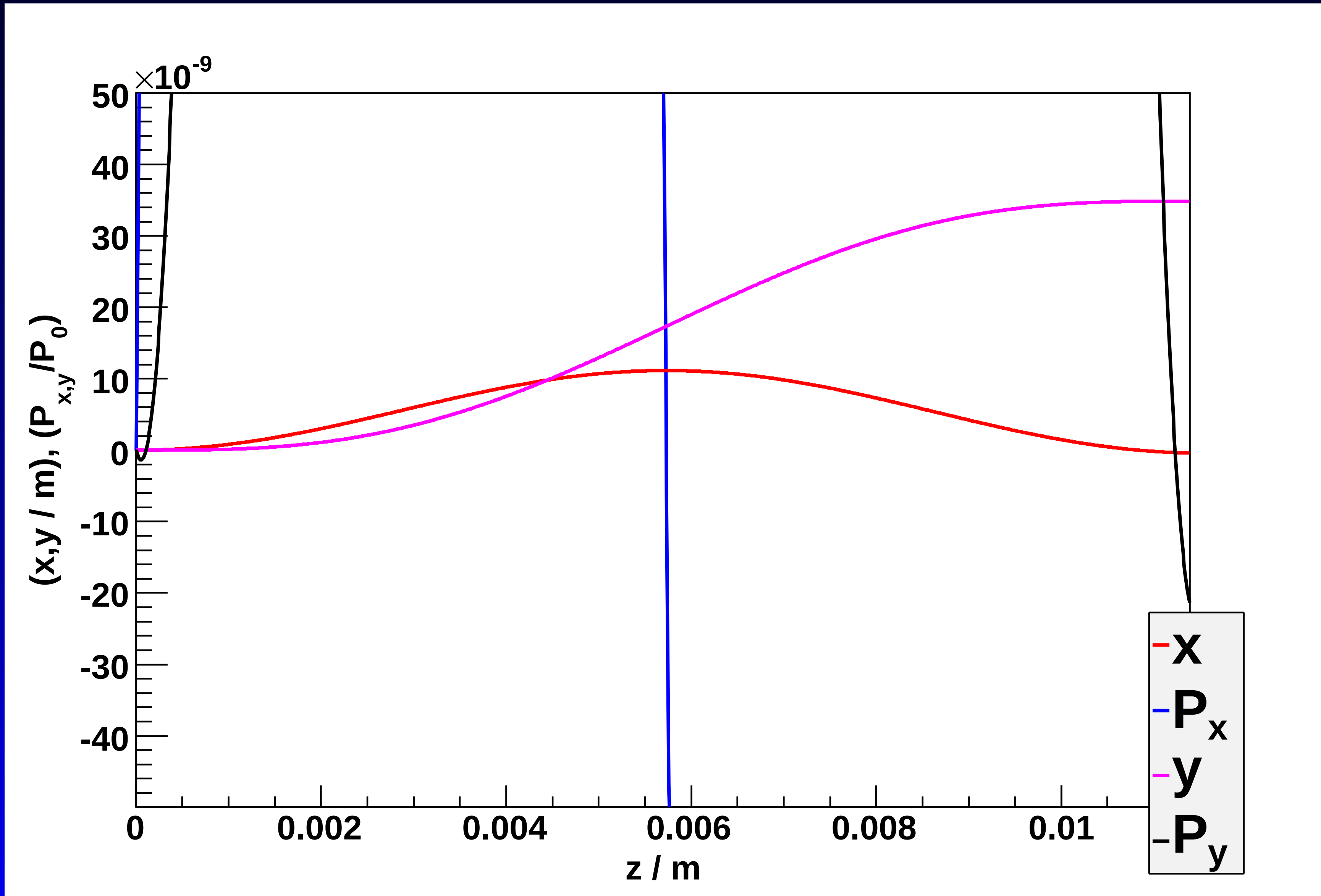
ILC Helical Undulator Field Map



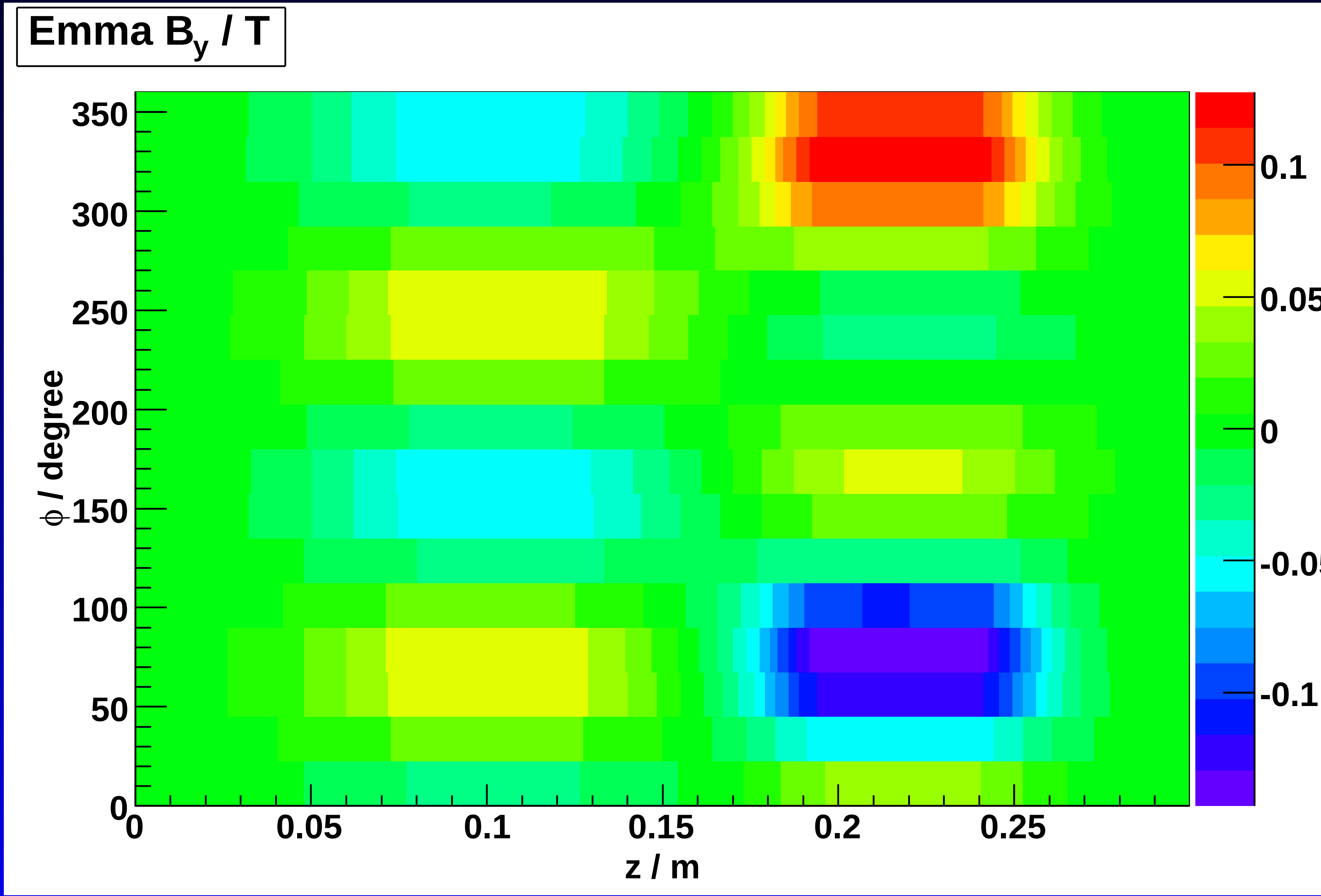
ILC Helical Undulator Tracking



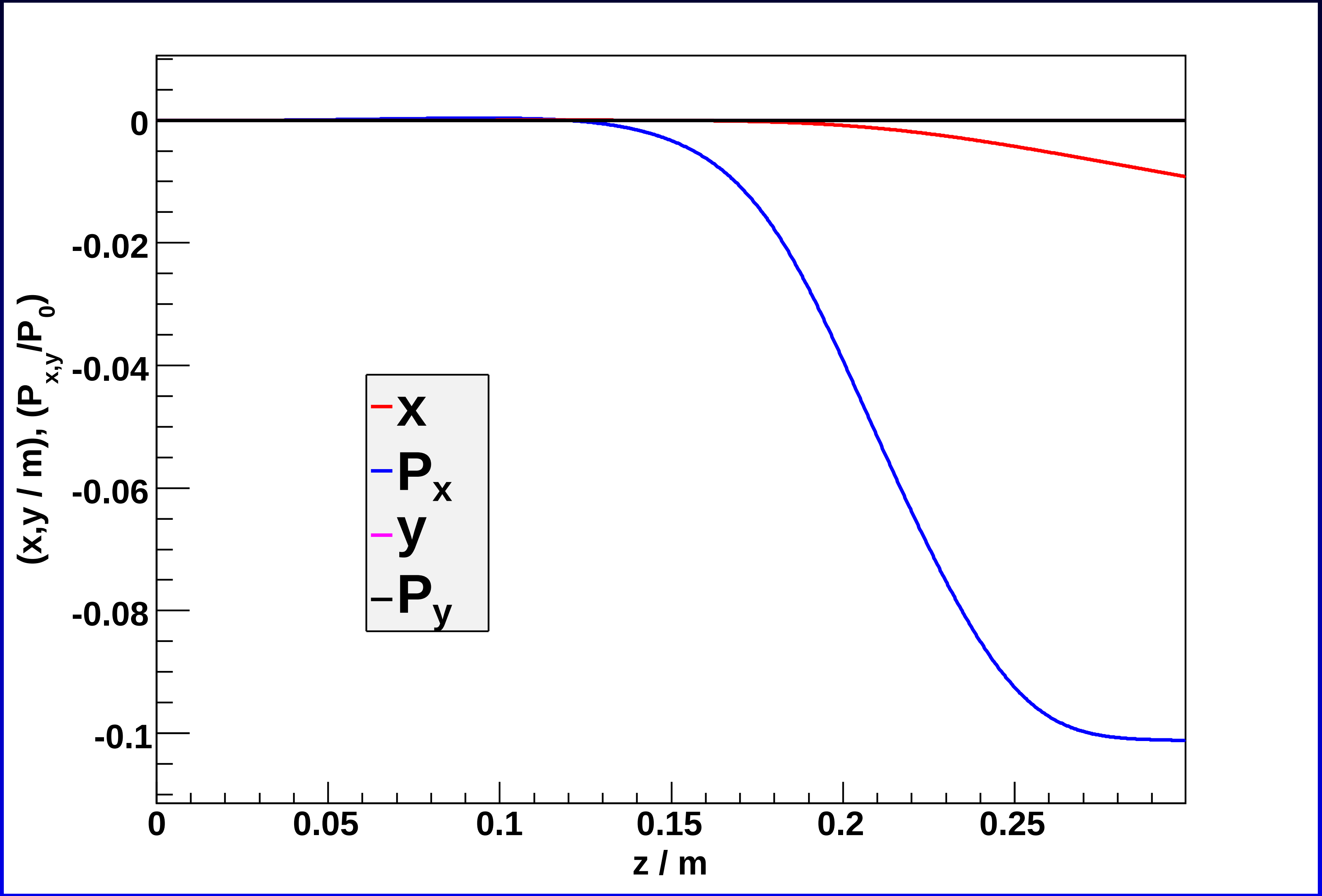
ILC Helical Undulator Tracking



Emma Field Map



Emma Tracking



To Do...

- Comparison of EMMA Transfer Map with the work of Yoel, Stefan Tzenov
- Look at Applications to EMMA lattice
- Possibility of tying in the ILC undulator tracking code with SPUR (synchrotron radiation code)
- Continue to test and optimise the code.