Fast, accurate calculation of dynamical maps from magnetic field data using generalised gradients

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Abstract

Analytic descriptions of arbitrary magnetic fields can be calculated from the generalised gradients [1] of the on-axis field. Using magnetic field data, measured or computed on the surface of a cylinder, the generalised gradients can be calculated by solving Laplace's equation to find the three-dimensional multipole expansion of the field within the cylinder. After a suitable transformation, this description can be combined with a symplectic integrator [2] allowing the transfer map to be calculated. A new tracking code is under development in C++, which makes use of a differential algebra class to calculate the transfer map. The code has been heavily optimised to give a fast, accurate calculation of the transfer map for an arbitrary field. The multipole nature of the field description gives additional insights into fringe-field and pseudo-multipole effects and allows a deeper understanding of the beam dynamics.

Magnetic field descriptions from generalised gradients

The magnetic vector potential for any field can be derived from a cylindircal harmonic expansion and described as:

$$A_\phi = 0$$

$$A_\rho = \sum_{m=1}^{\infty} \frac{\cos(m\phi)}{m}\rho\frac{\partial}{\partial z}\psi_{\omega,s} - \frac{\sin(m\phi)}{m}\rho\frac{\partial}{\partial z}\psi_{\omega,c}$$

$$A_z = \sum_{m=1}^{\infty} -\frac{\cos(m\phi)}{m}\rho\frac{\partial}{\partial \rho}\psi_{\omega,s} + \frac{\sin(m\phi)}{m}\rho\frac{\partial}{\partial \rho}\psi_{\omega,c}$$

where,

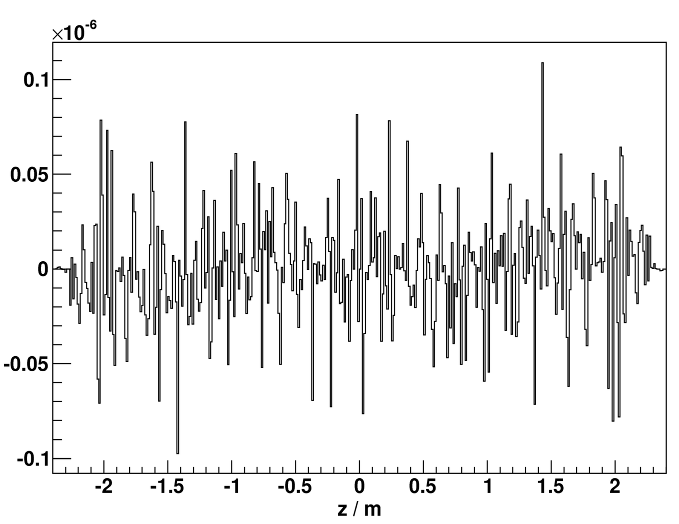
$$\psi_m(\rho,z) = \sum_{l=0}^{\infty}(-1)^l \frac{|m|!}{2^{2l}l!(l+|m|)!}\rho^{(2l+m)}C_m^{[2l]}(z)$$

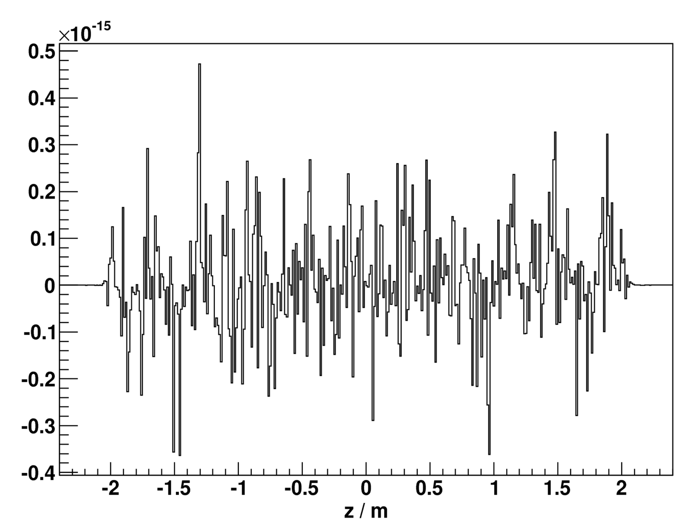
and,

$$C_{m,s}^{[l]}(z) = \frac{i^l}{2^m m!}\int_{-\infty}^{\infty} dk \exp(ikz)\frac{k^{l+m-1}}{I'_m(kR)}\hat{b}_m$$

$$C_{m,c}^{[l]}(z) = \frac{i^l}{2^m m!}\int_{-\infty}^{\infty} dk \exp(ikz)\frac{k^{l+m-1}}{I'_m(kR)}\hat{a}_m$$

where $C^{[n]}_m$ are the on-axis generalised gradients. The coefficients a_m and b_a are the Fourier coefficients of the radial component of the magnetic field on the surface of a cylinder of radius ρ. Using this technique an expression for the magnetic field can be found at any point within the volume of the cylinder [1],[4]. This expression is analytic in the transverse plane, and has only a z-dependence given by the generalised gradients. Furthermore the algorithm has a smoothing quality – numerical inaccuracies in the original field are smeared out.





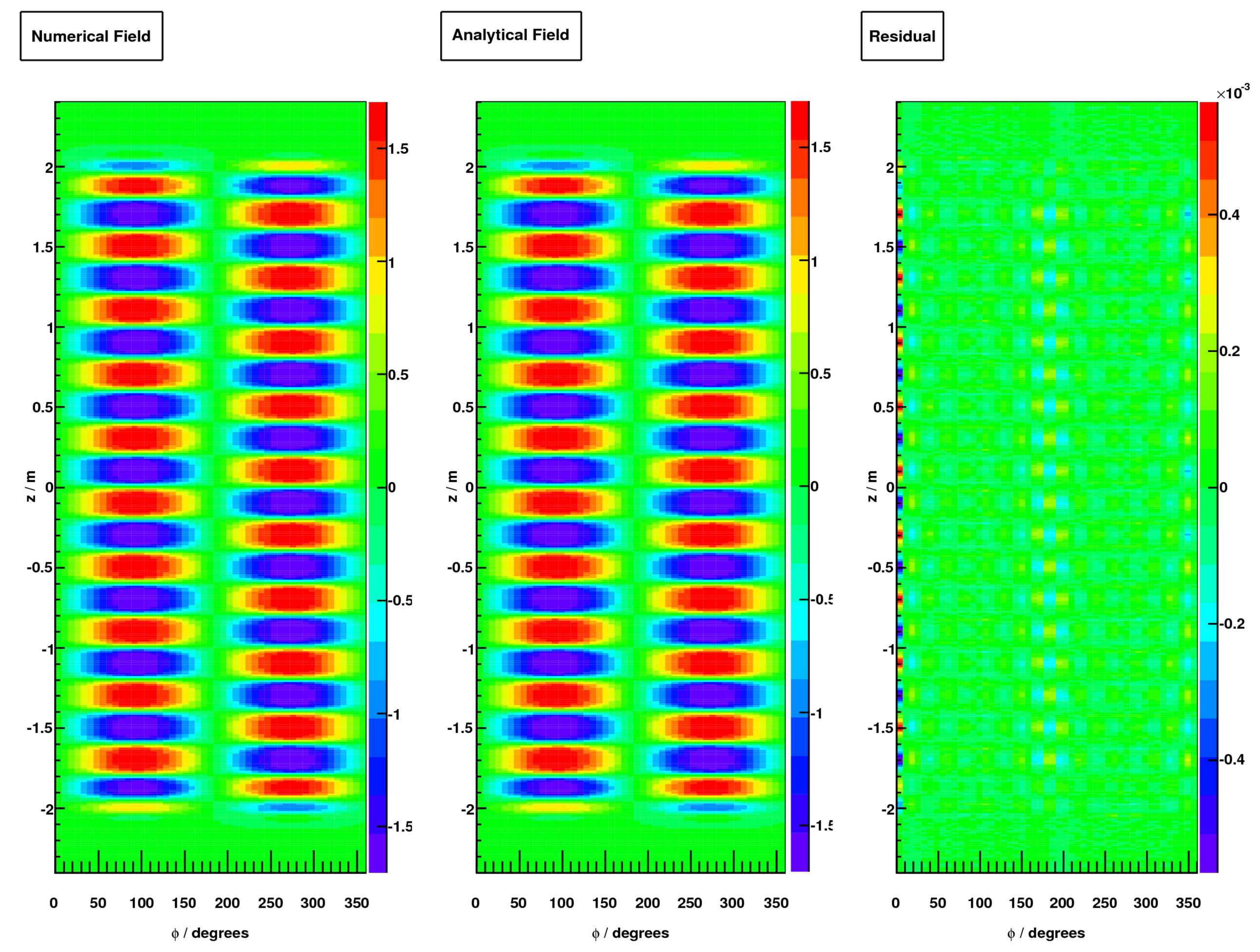
Field smoothing: the quantity :

$$\oint_{\partial S} B \cdot dl$$

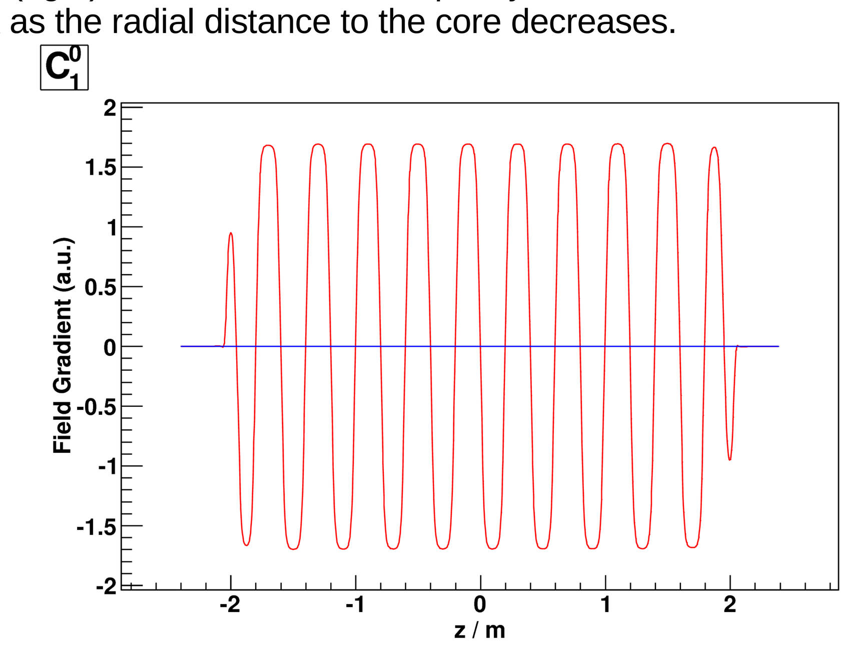
, the line integral of the magnetic field around a closed surface (in this case, in the xy plane) should sum to zero according to Maxwell's equations (Ampere's circuital law). The plot on the left shows the integral, evaluated at different z coordinates, for the initial numerical field map. The non-zero values are due to numerical inaccuracies in the field map. The plot on the left shows the integral for the final analytical map. The numerical inaccuracies have been reduced by ~9 orders of magnitude - close to the computers precision.

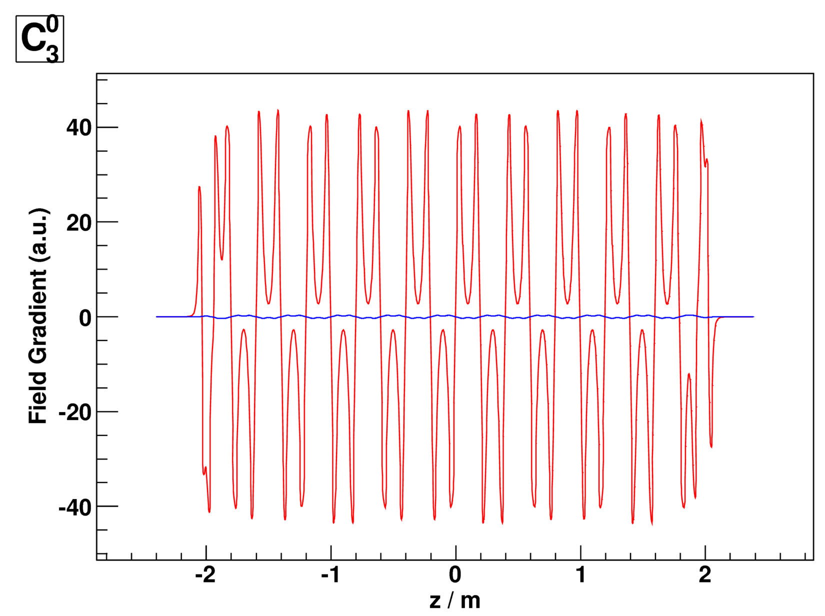
The on-axis generalised gradients

The on-axis generalised gradient $C^{[n]}_{m,\alpha}$ is the nth derivative of the generalised gradient $C^{[0]}_{m,\alpha}$ where m indicates the order of the multipole expansion (i.e. m=0 corresponds to a solenoid component, m=1 corresponds to the dipole component, m=2 is the quadrupole component etc.) and α is either c (the skew cosine component of the Fourier transform) of s (the normal sine component of the Fourier transform). An analytic description of the magnetic field can be derived in terms of these generalised gradients [1].



Shown above is the original numerically calculated field map (Bρ on the surface of a cylinder) for an wiggler [5] section (left), the reconstructed field using the generalised gradients (centre) and the residual between the two maps (right). The maximum discrepancy is 2.10^{-4} Tesla at the surface of the cylinder. This discrepancy will shrink as the radial distance to the core decreases.





The generalise gradients $C^{[0]}_1$ (top and $C^{[0]}_3$ (bottom}. Both normal red) and skew (blue) components are shown. The $C^{[0]}_1$ (dipole) generalised gradients correspond to the on-axis field components B_x (skew) and B_y (normal). The $C^{[0]}_3$ generalised gradients, correspond to the sextupole component.

A Second order symplectic integrator

A second order symplectic integration scheme has been implemented [2] which calculates the Lie map to transform the canonical phase space vector (x,px,y,py,x,δ) from an initial to a final state for an integration step Δσ.

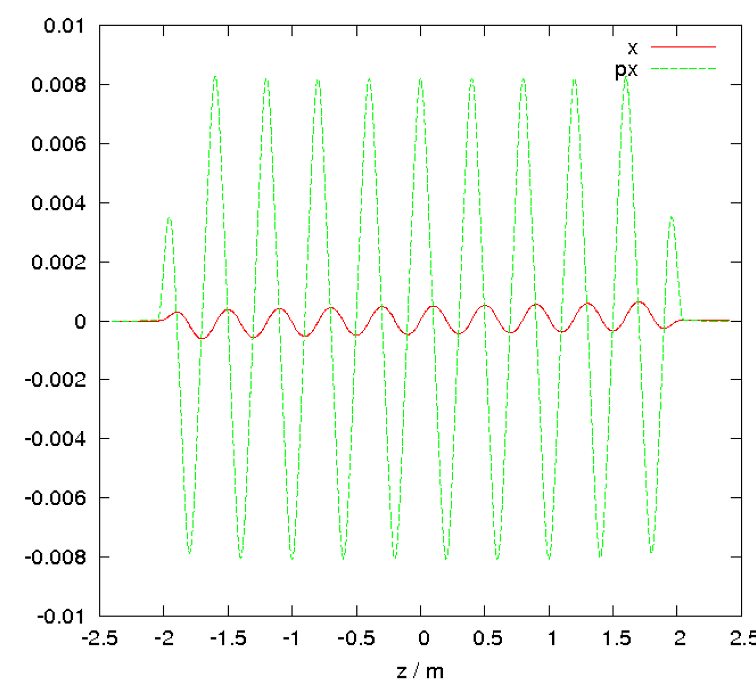
$$\begin{aligned}\mathcal{M}(\Delta\sigma) = {} & \exp\left(:-\frac{\Delta\sigma}{2}P_z:\right)\exp\left(:-\frac{\Delta\sigma}{2}a_z:\right) \\ & \exp\left(:-\frac{\Delta\sigma}{2}\left(-\delta+\frac{P_x^2}{2(1+\delta)}\right):\right)\mathcal{A}_y \\ & \exp\left(:-\Delta\sigma\frac{P_y^2}{2(1+\delta)}:\right)\mathcal{A}_y^{-1} \\ & \exp\left(:-\frac{\Delta\sigma}{2}\left(-\delta+\frac{P_x^2}{2(1+\delta)}\right):\right) \\ & \exp\left(:-\frac{\Delta\sigma}{2}a_z:\right)\exp\left(:-\frac{\Delta\sigma}{2}P_z:\right)\end{aligned}$$

Where

$$a_{x,y,z}(x,y,z) = qA_{x,y,z}(x,y,z)$$

$$\mathcal{A}_y = \exp\left(:-\int a_y(x,y,z)dy:\right)$$

The canonical phase vector was integrated over 10,000 steps to track their evolution as a 5 GeV electron traversed the wiggler field.



Conclusion

A new C++ code has been written to describe arbitrary magnetic fields and calculate the evolution of the dynamical variables of a charged particle within such a field has been developed. The code is designed to provide a fast, accurate method of describing charged particle dynamics. In the given example, for a 4.8 m wiggler magnet, the field was interpolated at 49x8192=401,408 ponts on the surface of a cylinder. The field was used to calculate numerically hte generalised gradients for the multipole components up to the 12th pole: this provides and analytical description of the transverse field. Finally the field description was used to integrate analytically and numerically the evolution of a phase space vector (to second order in the Hamiltonian and sixth order in the field description) over the length of the magnet using 10,000 integration steps. The whole calulation took under 8 minutes on a 2.66 GHz processor, although once the transfer matrices are calculated, numerical tracking takes a matter of seconds. The code is modular, so additional functionality can easily be added. Further examples of the applications of this code and an example of a module to calculate synchrotron radiation are described in reference [3].

References

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