A study of the maximum entropy technique for phase space tomography

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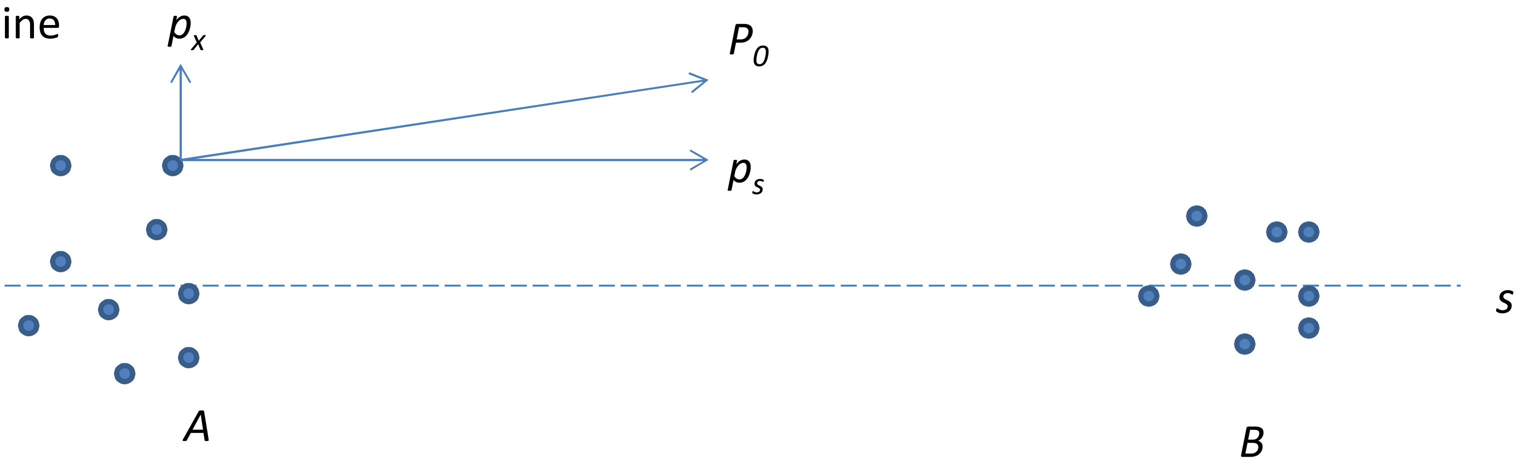
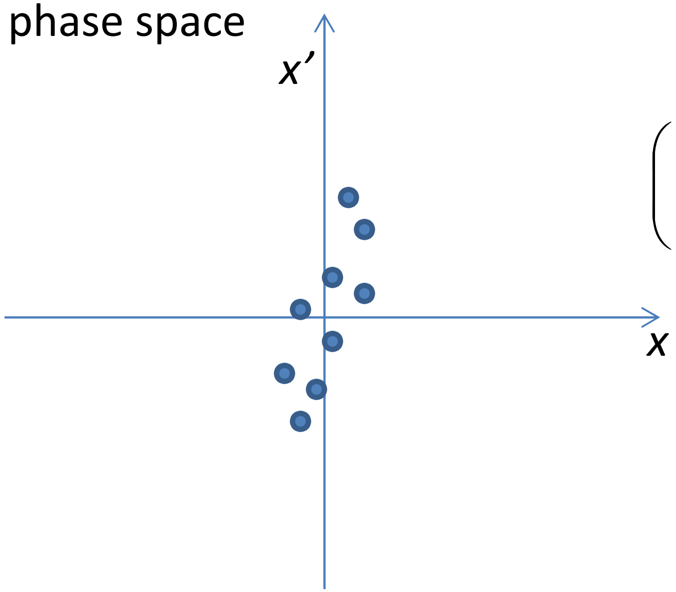
Phase space

beamline



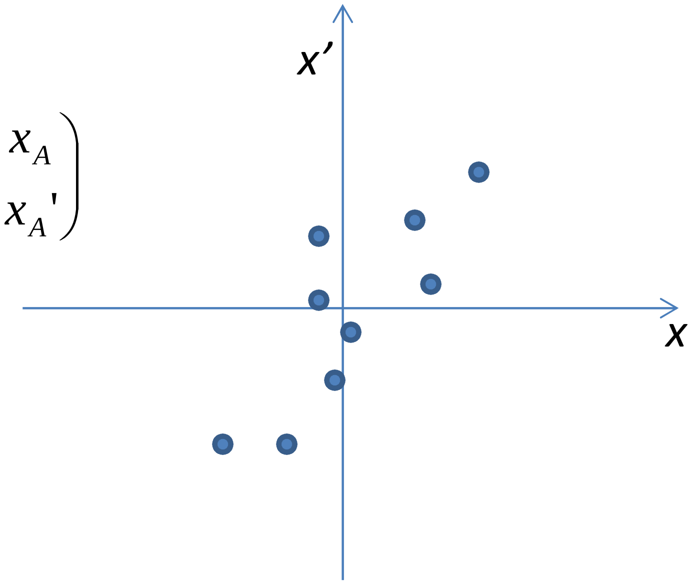
normalised momentum: $x' = dx/ds = p_x/P_0$

phase space

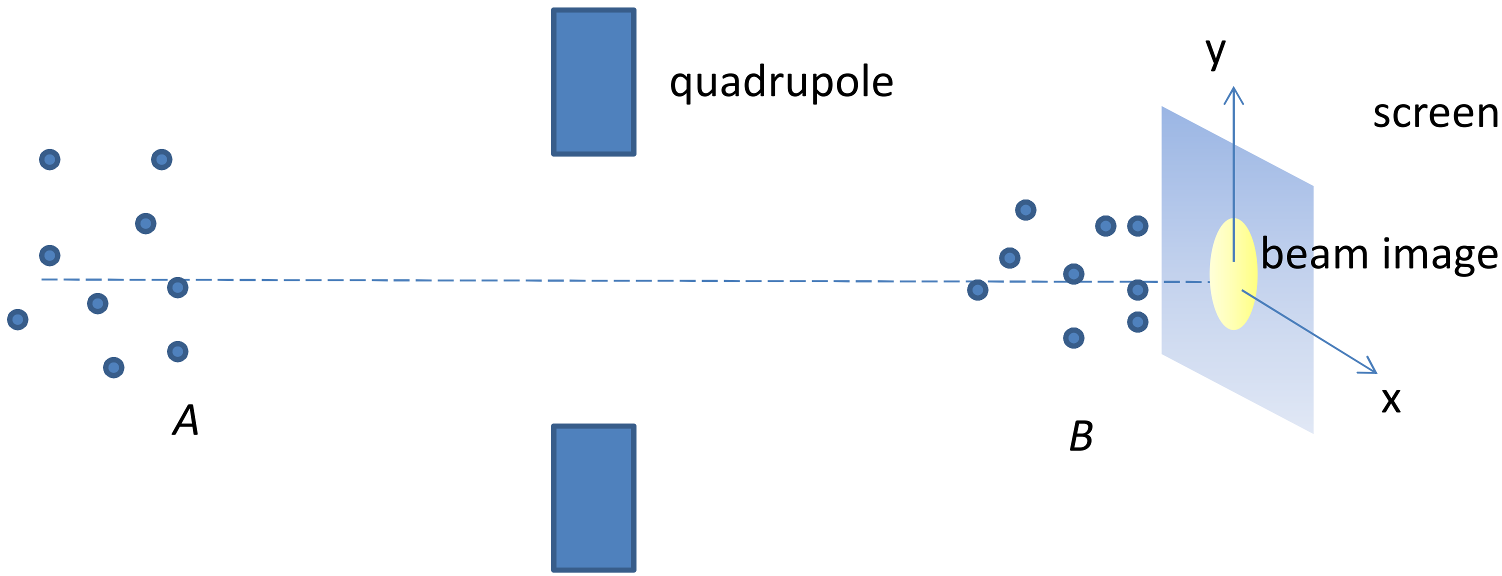


linear mapping

$$\begin{pmatrix} x_B \\ x_B' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x_A \\ x_A' \end{pmatrix}$$



Experiment

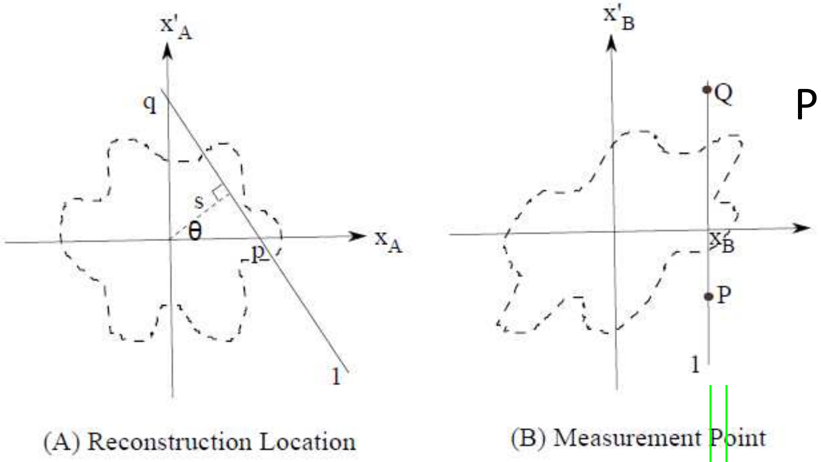


1. Take a photograph of the beam image at B.
2. Change quadrupole strength.
3. Take another photo.
4. Repeat for 100 – 200 times.
5. Use a code to reconstruct the distribution of particles in phase space at A.

Tomography

Linear mapping is a geometrical transformation, like rotation, shearing, stretching, etc.

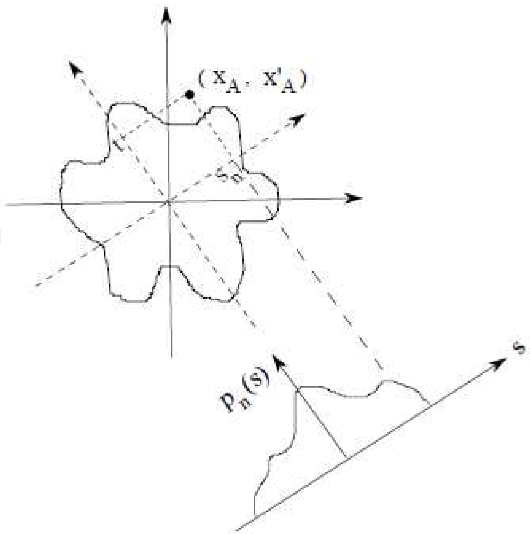
So a rotated line at A maps can map to a vertical line at B.

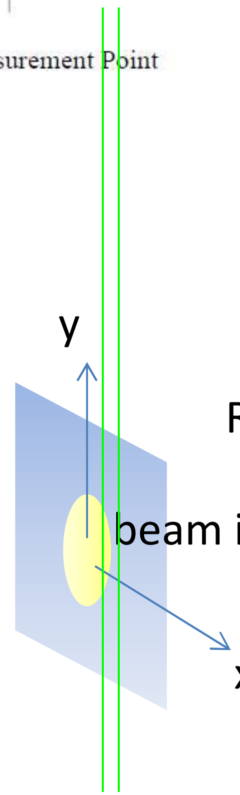


Phase space

The same interval of x in real and phase space contains the same number of particles

A projection along the rotated direction at A can be calculated from beam image at B.

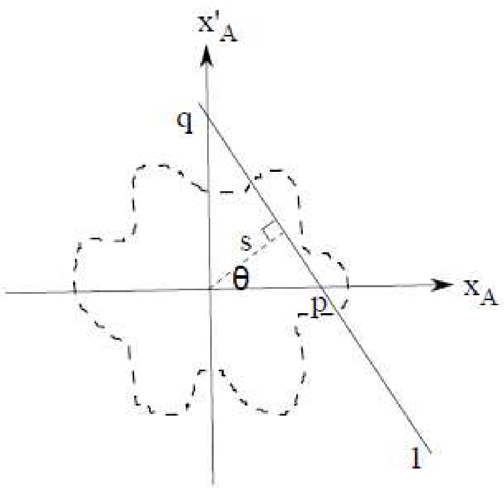




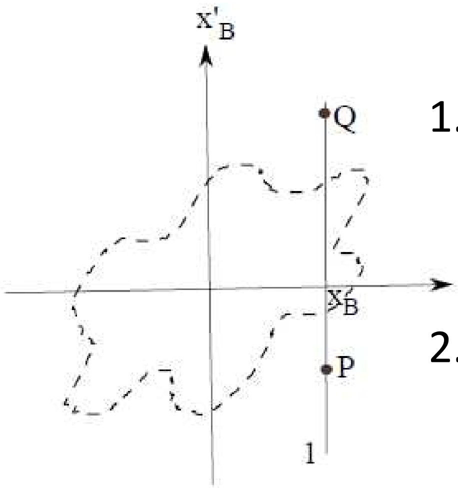
Real space

beam image

Reconstruction

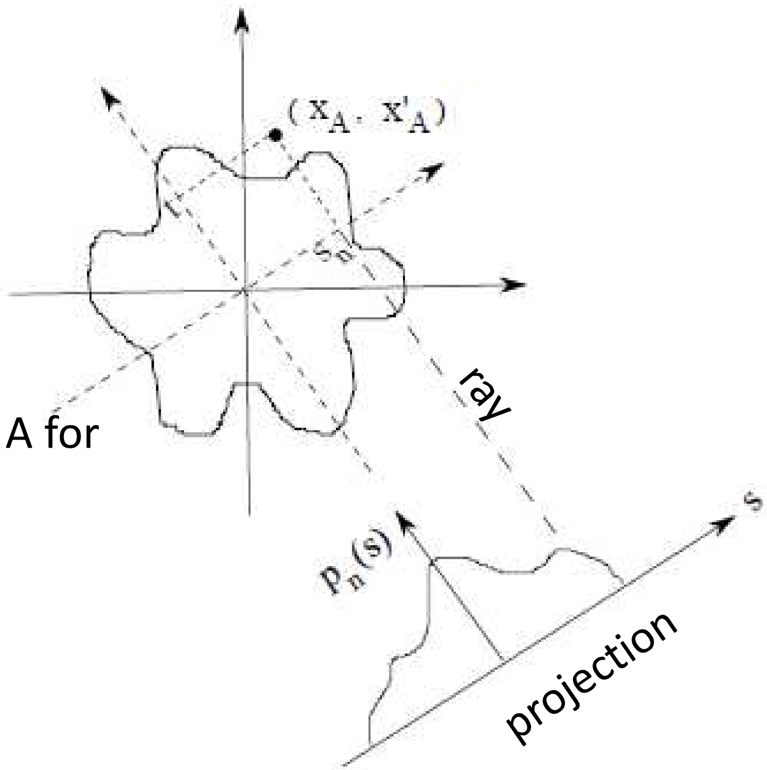


(A) Reconstruction Location

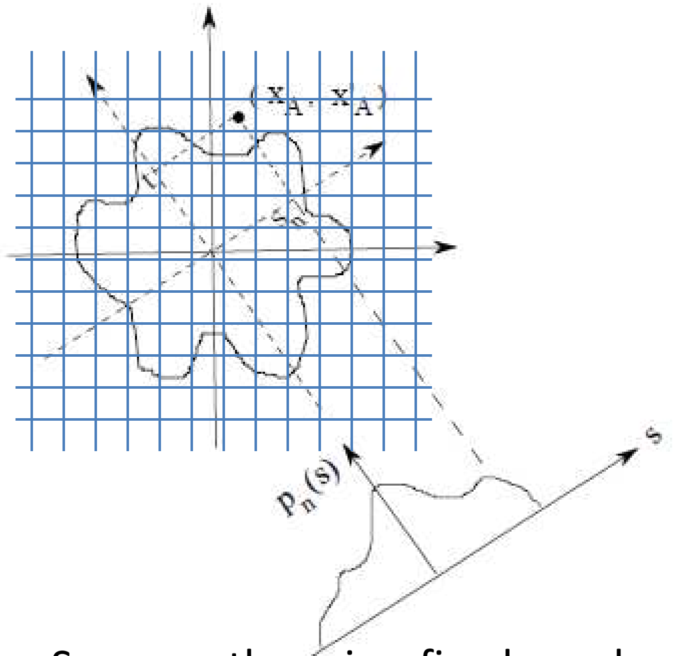


(B) Measurement Point

1. Each quadrupole strength corresponds to a geometrical transformation and a certain rotation angle.
2. Changing quadrupole strengths during measurement means changing rotation angle at A.



1. From the beam images collected at B, projections at A for a range angles can be computed.
2. A standard reconstruction technique is used to reconstruction the distribution, e.g. Filtered Back Projection (FBP) used in CAT scan.



Maximum Entropy Technique (MENT)

What if we have only a few projections? Can we make a guess of the distribution?

Yes! MENT can give us the most likely answer.

Problem statement:

Suppose there is a fixed number of particles, distributed among squares elements. Suppose a measured projection must agree with that on the distribution. Suppose that as long as they obey these two constraints, the particles are equally like to be in any square. Find the most likely distribution among the squares.

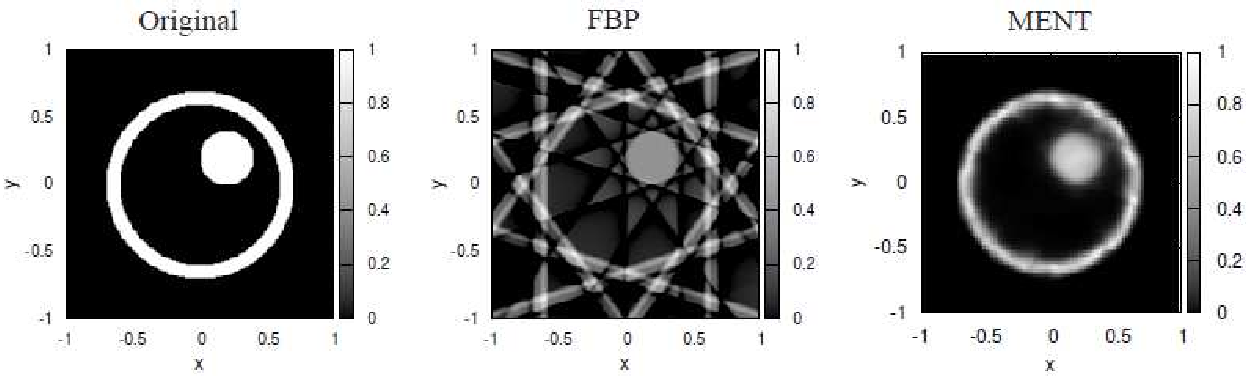
Solution: The solution is obtained by finding the distribution in which the particles can have the greatest possible number of arrangements among the levels (or squares). The method is Lagrange multipliers. The answer is:

The most likely distribution f(x,x') is a product of functions. Each function is a function of one variable. This variable is a coordinate of a measured projection (*s* in the figure).

With this information and the measured projections, f(x,x') can be computed iteratively.

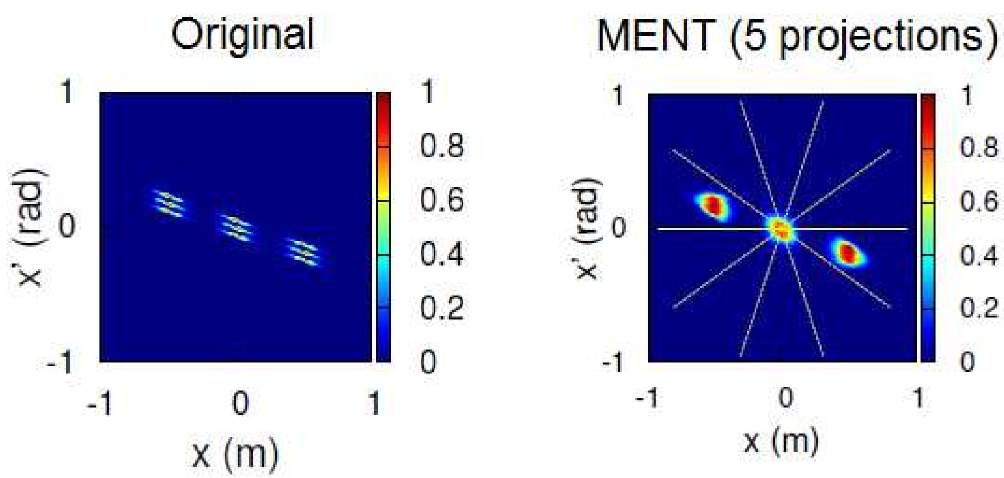
Problem

MENT seems very good because with just a few projections it can give results that look very nice where FBP is completely hopeless.



But with so few projections, won't it miss out a lot of details?

Definitely!



Ray direction

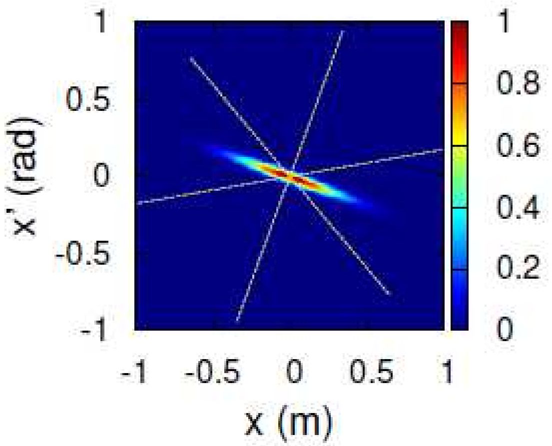
There is one way we can improve things – with a suitable choice of angles.

Normalised Phase Space

In linear beam dynamics, mappings can be transformed to normalised phase space (α = 0, β = 1 m) to simplify calculations.

Transforming rays from (x,x’) to normalised phase space shows that the distribution may not be properly sampled if it is quite narrow.

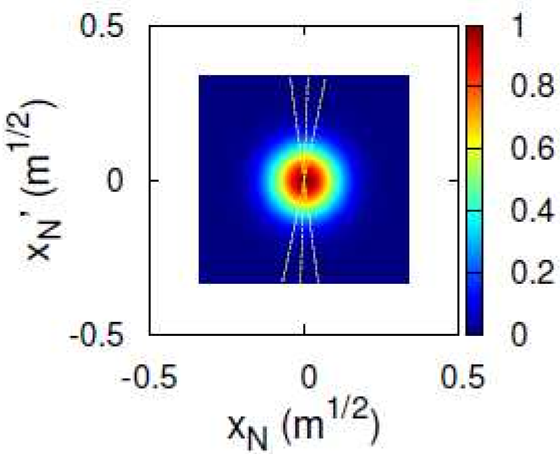
(x, x’) phase space



mapping

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} x_N \\ x'_N \end{pmatrix}$$

normalised phase space

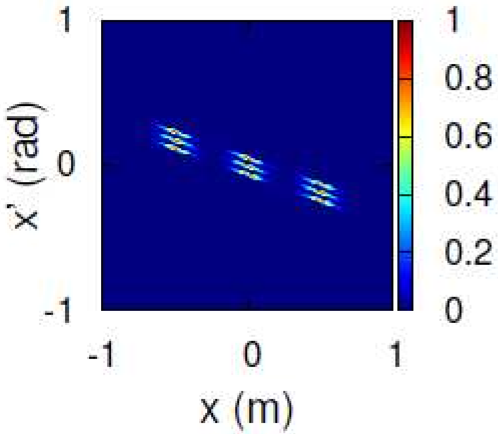


It makes more sense to sample at equal intervals of angles in normalised phase space instead of in (x,x’) phase space.

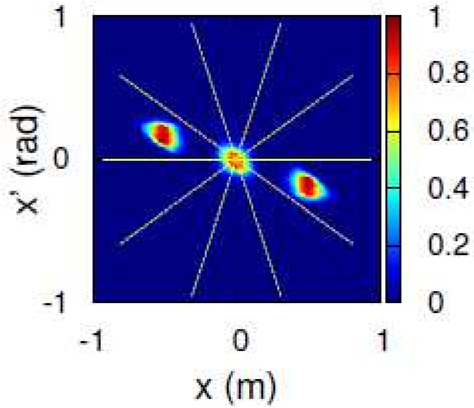
Simulation

Choosing equal intervals of angles in normalised phase space can indeed improve the results.

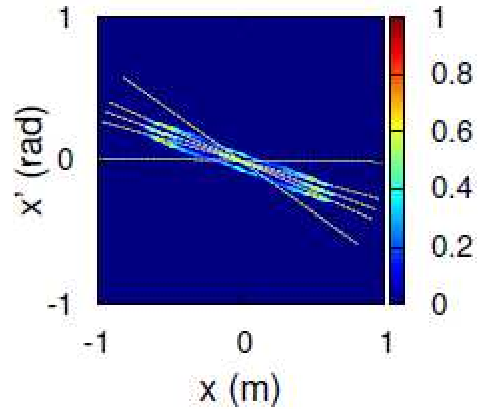
Original



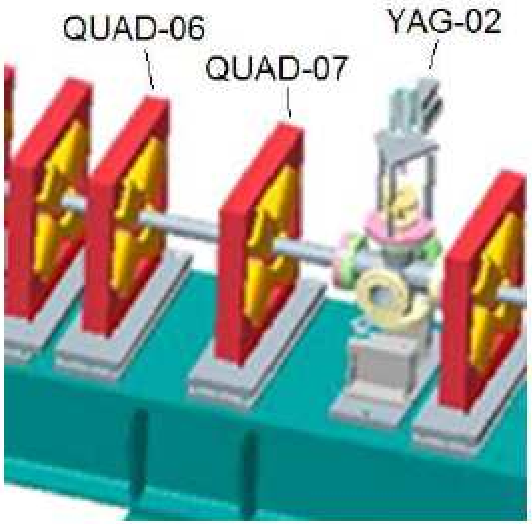
Equal angular intervals in (x,x') space



Equal angular intervals in normalised space



Measurement

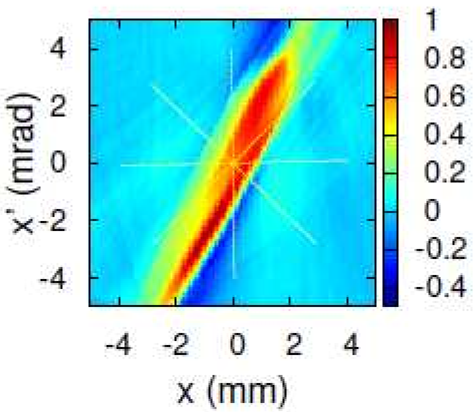


ALICE tomography section

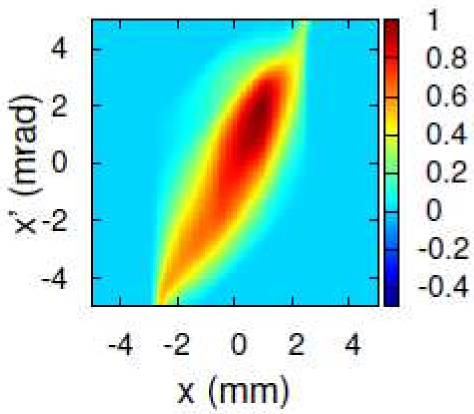
In practice, the Twiss parameters must first be measured before we can determine the required angles in normalised phase space.

The results below are obtained by selecting appropriate angles from a previous FBP measurement that recorded about 150 angles.

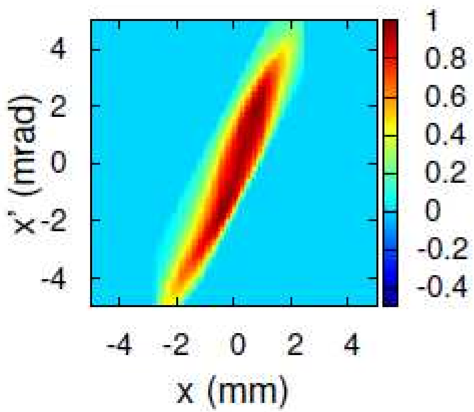
"Original"
(actually FBP result)



Equal angular intervals in (x,x') space



Equal angular intervals in normalised space



Conclusion

ALICE, PITZ, FLASH, TRIUMF, PSI and SNS all have beamline components designed to do MENT. A common setup is a number of screens at different locations to vary the drift space transfer matrix, instead of changing the strength of quadrupole.

MENT requires few projections and is suitable for simple setup and fast measurement. But this also means less information and more guesswork, which our work hopes to address.

FBP requires computer control of quadrupole strengths and camera recording to acquire 100 or more images rapidly. The reconstructed results could have long, confusing lines in the background which is absent in MENT.

The two methods can confirm and complement each other. (There are also other methods.)

Further work to check if this work is practical and reliable would be nice, but it depends on the availability of ALICE or EBTF.

This work has been published. You will find more details here:

K. M. Hock and M. G. Ibison, "A study of the maximum entropy technique for phase space tomography", Journal of Instrumentation, **8** (2013) P02003.