Lecture 16

- Gravitation at Earth's surface
  - Effects of rotation
  - Effects of oblation
- Orbital Motion
  - Centre of mass coordinates

Gravitation at Earth's Surface

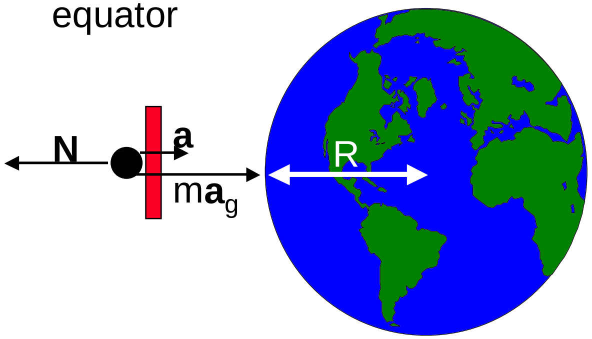
- Label the acceleration at the surface of a spherical non-rotating earth a_g. We have shown that (making downwards positive).

$$\begin{aligned} a_g &= \frac{GM}{R^2} \\ &= \frac{6 \cdot 673 \times 10^{-11} \times 5.977 \times 10^{24}}{\left(6371 \times 10^3\right)^2} \\ &= 9.826\,\mathrm{m\,s^{-2}} \end{aligned}$$

- Look now at more realistic scenario in which earth
  - rotates,
  - is oblate.

Gravitation at Earth's Surface cont.

- Consider first effect of rotation at equator



- Mass m on scale at equator. Scale exerts force N. Newton's 2nd Law tells us

$$\sum \mathbf{F} = ma$$

$$\Rightarrow ma_g - N = mR\omega^2$$

Gravitation at Earth's Surface cont.

- We interpret N as being due to accel. due to gravity g′ so:

$$N = ma_g - mR\omega^2$$

$$\Rightarrow mg' = ma_g - mR\omega^2$$

$$g' = a_g - R\omega^2$$

- Hence

$$g'_E = a_g - R\omega^2 = a_g - R\left(\frac{2\pi}{T}\right)^2$$

$$= 9\cdot 826 - \frac{6\cdot 371\times 10^6 \times (2\pi)^2}{(24\times 60\times 60)^2}$$

$$= 9\cdot 792\,\mathrm{m\,s^{-2}}$$

$$\Delta g'_E = 0\cdot 034\,\mathrm{m\,s^{-2}}$$

- At poles

$$g'_P = a_g \Rightarrow \Delta g'_P = 0$$

Gravitation at Earth's Surface cont.

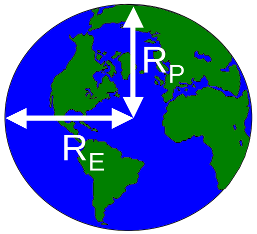
- Oblation, equatorial radius R_E, polar radius R_P, mean radius R

$$R_E = 6378 \cdot 2\,\text{km}$$

$$R_P = 6356.8\,\text{km}$$

$$R_E = R_P + 21 \cdot 4\,\text{km}$$

$$R = 6371\,\text{km}$$



- Influence on polar and equatorial accel. due to grav.

$$g_E'' = \frac{GM}{R_E{}^2} = 9.805\,\text{m s}^{-2}$$

$$\Delta g_E'' = 0 \cdot 021\text{ms}^{-2}$$

$$g_P'' = \frac{GM}{R_P{}^2} = 9.870\,\text{m s}^{-2}$$

$$\Delta g_P'' = -0 \cdot 044\,\text{m s}^{-2}$$

Gravitation at Earth's Surface cont.

- Adding the two effects:

$$g_E = 9 \cdot 77\,\text{m s}^{-2}$$

$$g_P = 9 \cdot 87\,\text{m s}^{-2}$$

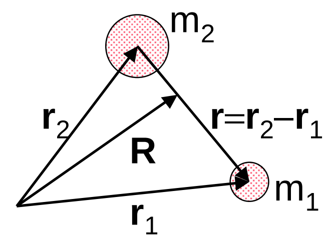
- In addition measurements will be affected by altitude and local variations in thickness and density of crust. This latter effect is used when prospecting for oil. Look for changes in accel. due to gravity ("anomalies") of order 10 milligals, where 1 gal is 1 cm s^{-2}.

Centre-of-Mass Coordinates

- Want to consider orbits such as moon around earth. Two body problem, earth and moon influence one another's motion, makes equations difficult.
- Recall discussion of rigid bodies, Motion can be considered to be translational motion of point with mass of entire body at c. of m. plus rotation. Can we use this idea to help us discuss orbits?

Centre of mass coords cont.

- Consider two masses interacting via gravitational force.



- Centre of mass position given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} \quad \text{where } M = m_1 + m_2$$

- Equations of motion of masses

$$\mathbf{F}_1 = m_1 \frac{d^2\mathbf{r}_1}{dt^2} \qquad (1)$$

$$\mathbf{F}_2 = m_2 \frac{d^2\mathbf{r}_2}{dt^2} \qquad (2)$$

Centre of mass coords cont.

- Add (1) and (2), use $\mathbf{F}_1 = -\mathbf{F}_2$

$$m_1 \frac{d^2\mathbf{r_1}}{dt^2} + m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{F}_1 + \mathbf{F}_2 = 0 \qquad (3)$$

- Determine $\mathbf{r}_1$ in terms of $\mathbf{R}$ and $\mathbf{r}$

$$M\mathbf{R} = m_1\mathbf{r_1} + m_2\mathbf{r_2}$$
$$= m_1\mathbf{r_1} + m_2(\mathbf{r} + \mathbf{r}_1)$$
$$\therefore \mathbf{r_1} = \mathbf{R} - \frac{m_2}{M}\mathbf{r} \quad \left(\text{also } \mathbf{r}_2 = \mathbf{R} + \frac{m_1}{M}\mathbf{r}\right)$$

- Substitute for $\mathbf{r}_1$ and $\mathbf{r}_2$ in (3)

$$m_1 \frac{d^2\mathbf{R}}{dt^2} - \frac{m_1 m_2}{M}\frac{d^2\mathbf{r}}{dt^2} +$$
$$m_2 \frac{d^2\mathbf{R}}{dt^2} + \frac{m_1 m_2}{M}\frac{d^2\mathbf{r}}{dt^2} = 0$$
$$\Rightarrow M\frac{d^2\mathbf{R}}{dt^2} = 0$$

Centre of mass coords cont.

- As we expect: no external forces acting so no acceleration. C. of m. moves with constant velocity.
- Now subtract (1) and (2)

$$m_2 \frac{d^2\mathbf{r_2}}{dt^2} - m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \mathbf{F}_2 - \mathbf{F}_1 = 2\mathbf{F}$$

- Express in terms of $\mathbf{R}$ and $\mathbf{r}$

$$m_2 \frac{d^2\mathbf{R}}{dt^2} + \frac{m_1 m_2}{M}\frac{d^2\mathbf{r}}{dt^2} -$$
$$m_1 \frac{d^2\mathbf{R}}{dt^2} + \frac{m_1 m_2}{M}\frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{F}$$

Centre of mass coords cont.

- We know $\frac{d^2\mathbf{R}}{dt^2} = 0$ so we have

  $$\mathbf{F} = \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\mathbf{r}}{dt^2} = \mu\frac{d^2\mathbf{r}}{dt^2}$$

  where we have defined the reduced mass

  $$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

- We have simplified the picture. Two body problem has become one body problem in centre of mass frame.