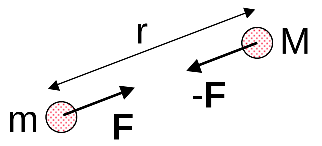
Lecture 15

- Newton's Law of Gravitation
- Gravitational Potential
  - Potential and force outside spherical shell
  - Potential and force inside spherical shell
- Measuring the Gravitational Constant
- Mass of the Earth

Newton's Law of Gravitation

- In 1665 Newton proposed that the force between point masses m and M separated by r is

$$F = -G\frac{mM}{r^2}$$



- The force is attractive, that on m being towards M and vice versa.
- This force acts between all masses in the universe.
- The gravitational constant has value

$$G = 6\cdot 67\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$$
$$= 6\cdot 67\times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$$

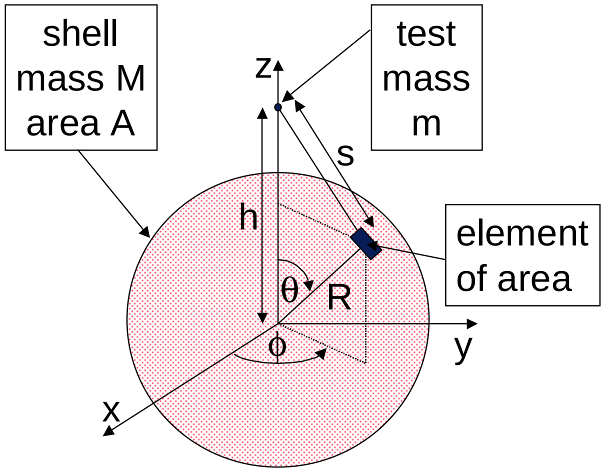
Gravitational Potential

- From relationship between force and potential can calculate gravitational potential.

$$U = -\int F\,dr$$
$$= -\int -\frac{GmM}{r^2}\,dr$$
$$= -\frac{GmM}{r}$$

- Now study the gravitational potential due to a spherical shell.

Gravitational potential cont.



$$\text{Surface density } \sigma = \frac{M}{A} = \frac{M}{4\pi R^2}$$

$$s^2 = (h - R\cos\theta)^2 + (R\sin\theta)^2$$
$$= h^2 - 2Rh\cos\theta + R^2$$
$$\Rightarrow 2s\,ds = 2Rh\sin\theta\,d\theta$$

Gravitational potential cont.

Sum potential due to interaction of test mass with all mass elements in shell

$$U = -\int_M \frac{Gm}{s}\,dm' = -Gm\int_A \frac{1}{s}\frac{dm'}{dA}\,dA$$

$$= -Gm\sigma\int_A \frac{1}{s}\,dA$$

Recall calc. of moment of inertia of sphere, element of volume in spherical polar coords.

$$dV = r^2 \sin\theta \, d\phi \, d\theta \, dr$$

So element of area is

$$dA = r^2 \sin\theta \, d\phi \, d\theta$$

Integral becomes

$$U = -Gm\sigma\int_0^{2\pi}\int_0^{\pi} \frac{R^2 \sin\theta \, d\phi \, d\theta}{s}$$

Gravitational potential cont.

Integrate over ϕ*, change variables to* s *and* ds

$$U = -2\pi Gm\sigma\int_{s_0}^{s_\pi} \frac{R^2 \sin\theta}{s}\frac{s\,ds}{Rd\sin\theta}$$

where $s_0 = +\sqrt{h^2 - 2Rh + R^2} = h - R$

and $s_\pi = +\sqrt{h^2 + 2Rh + R^2} = h + R$

Hence left with

$$U = -\frac{2\pi RGm\sigma}{h}\int_{h-R}^{h+R} ds$$

$$= -\frac{2\pi RGm\sigma}{h}\{(h`+R) - (h - R)\}$$

$$= -\frac{4\pi R^2 Gm\sigma}{h} = -\frac{4\pi R^2 Gm}{h}\frac{M}{4\pi R^2}$$

$$= -\frac{GmM}{h}$$

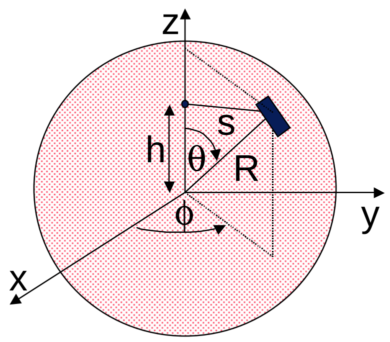
Gravitational potential cont.

- Same as expression if mass concentrated at centre of shell.
- Sphere consists of many concentric shells, hence also for sphere gravitational potential as though mass conc. at centre for objects outside sphere.
- Force derived from potential

  $$F = -\frac{d}{dh}U = -G\frac{Mm}{h^2}$$

  so above results apply also to gravitational force. (More difficult to calc. for vector force.)
- What about objects inside shell?

Gravitational potential cont.



Analysis proceeds exactly as before with one exception. Limits of integral over s *are now*

$$s_0 = +\sqrt{h^2 - 2Rh + R^2} = R - h$$

$$\text{and } s_\pi = +\sqrt{h^2 + 2Rh + R^2} = R + h$$

Gravitational potential cont.

So now we obtain

$$U = -\frac{2\pi RGm\sigma}{h}\int_{R-h}^{R+h} ds$$

$$= -\frac{2\pi RGm\sigma}{h}\{(R+h)-(R-h)\}$$

$$= -4\pi RGm\sigma = -4\pi RGm\frac{M}{4\pi R^2}$$

$$= -\frac{GmM}{R}$$

This does not depend on the position of the test mass within the shell.

Gravitational potential cont.

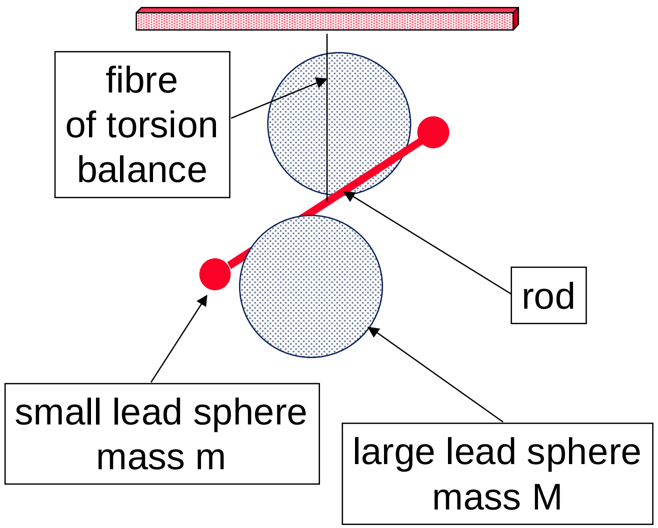
- Potential within shell constant. (This is the principle behind the Faraday cage which provides protection from electrical potential.)
- Force within shell

  $$F = -\frac{d}{dh}U = 0$$

- Hence a person going down a mine feels no gravitational force due to shell of earth at heights above his. Force due to rest of earth is as though concentrated at earth's centre.
- Newton worked all this out in 1665!

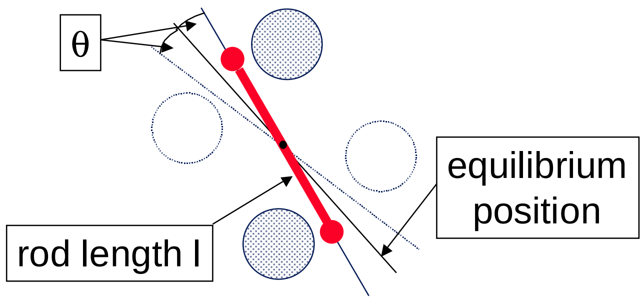
Measuring the Gravitational Constant

- Measure force between sphere's. First done by Cavendish, 1798.



Measuring G cont.

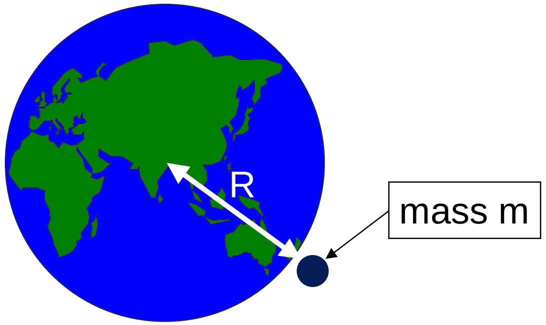
- Viewed from above



- Torque due to twisting of fibre
  $\tau_f = \kappa\theta$
- Torque due to gravity
  $$\tau_g = 2\frac{GMm}{d^2}\frac{l}{2} = \frac{GMml}{d^2}$$
- These balance so
  $$\kappa\theta = \frac{GMml}{d^2} \Rightarrow G = \frac{d^2\kappa\theta}{Mml}$$

Mass of Earth

- Knowing G and g can work out mass of earth. Assuming is uniform non-rotating sphere of mass M



$$g = \frac{F}{m} = -\frac{1}{m}\frac{GmM}{R^2} = -\frac{GM}{R^2}$$

$$\Rightarrow M = \frac{R^2 g}{G} = \frac{(6\cdot37\times10^6)^2\times9\cdot8}{6\cdot67\times10^{-11}}$$

$$= 6\cdot0\times10^{24}\ \text{kg}$$

Mass of earth cont.

- Density is

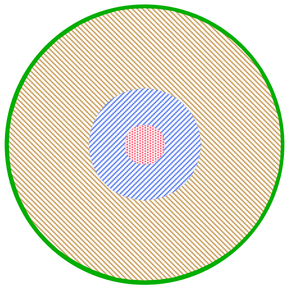
$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$= \frac{3\times6\cdot0\times10^{24}}{4\pi\times(6\cdot37\times10^6)^3}$$

$$= 5\cdot5\times10^3\ \text{kg}\,\text{m}^{-3}$$

- At surface we measure $\rho \approx 3\times10^3\ \text{kg}\,\text{m}^{-3}$ so centre of earth must be much more dense.

Mass of earth cont.



	R ($\times 10^6$ m)	ρ ($\times 10^3$ kg m^{-3})
Inner core	1.3	~13
Outer core	3.3	~11
Mantle	6.37	~4
Crust	6.37	~3

(Crust only 20 to 30 km thick)