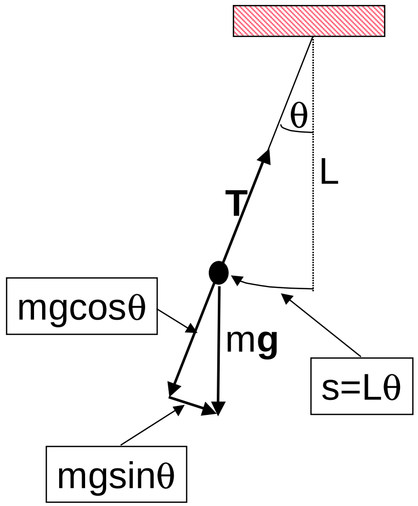
Lecture 13

- Simple Pendulum
- Angular SHM
- Physical Pendulum
- SHM and Uniform Circular Motion
- Complex Notation
- Damped SHM

The Simple Pendulum

- Consider small oscillations of a mass on the end of an inextensible light string.



Simple Pendulum cont.

- See restoring force is
  $F = -mg\sin\theta$
- Now using Taylor expansion of sine function about zero

$$\sin\theta = \sin(0) + (\theta - 0)\frac{\partial}{\partial\theta}\sin\theta\Big|_{\theta=0} + \frac{(\theta-0)^2}{2!}\frac{\partial^2}{\partial\theta^2}\sin\theta\Big|_{\theta=0} + \cdots$$

$$\Rightarrow \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\approx \theta \text{ for small } \theta$$

- So for small angles we have

$$F = -mg\theta$$

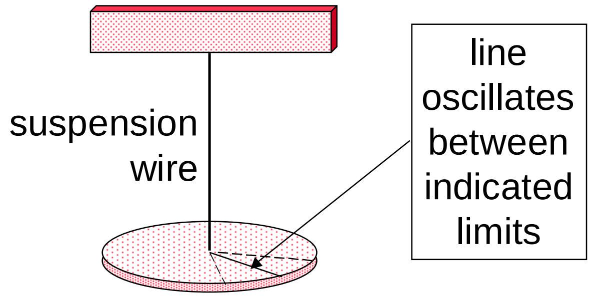
$$= -\frac{mg}{L}s$$

Simple pendulum cont.

- Cf. defining eqn. We have SHM. The spring constant is
  $k = \frac{mg}{L}$
- Hence period of oscillations is
  $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$
- Note, few real pendula are "simple". Simple pendula are not simple when the oscillations become large!

Angular SHM

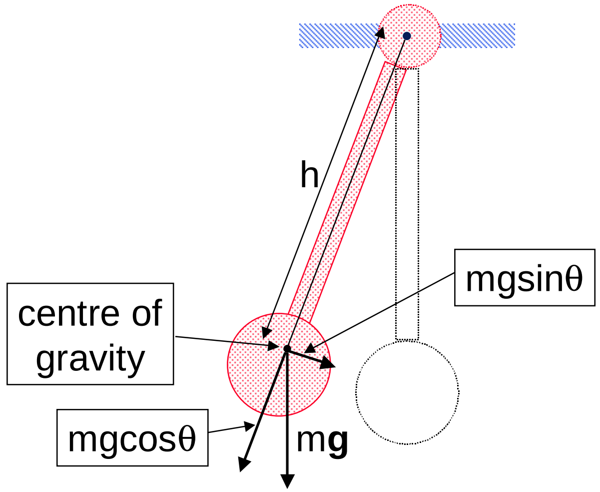
- Consider torsion pendulum



- Analogous to spring. Wire exerts torque when twisted through angle θ such that
  $\tau = -\kappa\theta$
- As in linear case get SHM, with period
  $$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

Physical Pendulum

- Consider more realistic pendulum.



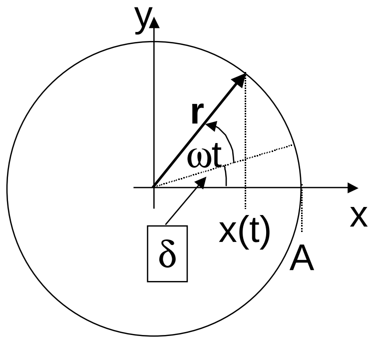
- If displaced by θ restoring torque appears.
  $$\tau = -mgh\sin\theta$$
  $$\approx -mgh\theta \text{ for small } \theta$$

Physical pendulum cont.

- Cf. defining eqn. for angular SHM
  $\kappa = mgh$
- Hence period of physical pendulum is
  $T = 2\pi\sqrt{\frac{I}{mgh}}$
- Apparent contrast to simple pendulum is dependence on mass, but beware!
  $I \approx mh^2$
  $\Rightarrow T \approx 2\pi\sqrt{\frac{h}{g}}$

SHM and Uniform Circular Motion

- SHM $x(t) = A\cos(\omega t + \delta)$
- Circular motion

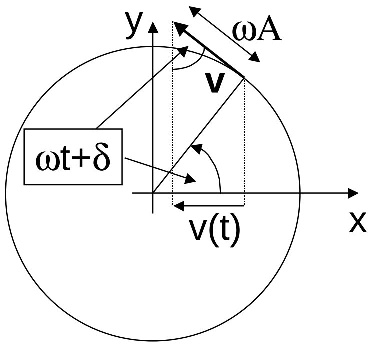


$$x(t) = A\cos(\omega t + \delta)$$

- SHM is the projection of uniform circular motion on a diameter of the circle about which the motion occurs. (Hence term “angular frequency”)

SHM and uniform circular motion cont.

- Now look at velocity, SHM
  $v(t) = -A\omega \sin(\omega t + \delta)$
- Cf. velocity in circular motion

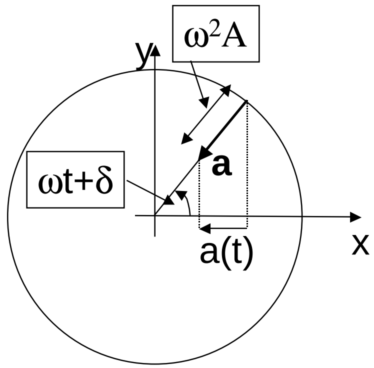


$$v(t) = -A\omega \sin(\omega t + \delta)$$

- Relationship holds also for velocity

SHM and uniform circular motion cont.

- Consider acceleration, SHM
  $a(t) = -A\omega^2 \cos(\omega t + \delta)$
- Cf. acceleration in circular motion



$$a(t) = -A\omega^2 \cos(\omega t + \delta)$$

- Relationship holds also for acceleration.

Complex Notation

- The differential equation we solved for SHM was

$$\frac{d^2}{dt^2}x + \omega^2 x = 0$$

- Look at complex function

$$x(t) = A\exp\left[i(\omega t + \delta)\right]$$

$$\Rightarrow \frac{d}{dt}x(t) = i\omega A\exp\left[i(\omega t + \delta)\right]$$

$$\Rightarrow \frac{d^2}{dt^2}x(t) = i^2\omega^2 A\exp\left[i(\omega t + \delta)\right]$$

$$= -\omega^2 A\exp\left[i(\omega t + \delta)\right]$$

$$= -\omega^2 x(t)$$

- So this is also solution to our equation.

Complex notation cont.

- Using the result

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

We see the following are equivalent

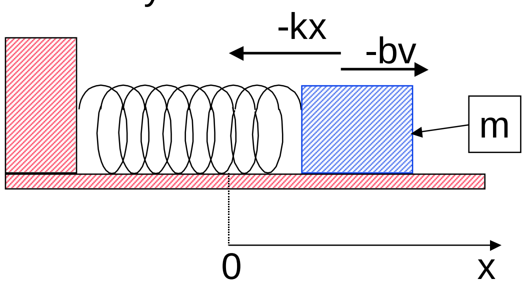
$$x(t) = A\cos(\omega t + \delta)$$

$$x(t) = \Re\left(A\exp\left[i(\omega t + \delta)\right]\right)$$

- Complex notation often brings significant advantages and will be used heavily in more advanced treatments of waves, in electrical circuit theory, in quantum mechanics...

Damped SHM

- A pendulum swinging in air loses mechanical energy and stops, the motion is said to be damped.
- Consider damping force proportional to velocity of motion.



- Newton's second law now gives

$$a = \frac{\sum F}{m} = -\frac{kx}{m} - \frac{bv}{m}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

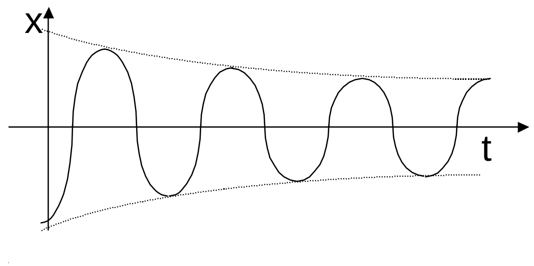
Damped SHM cont.

- If damping small ie. $\frac{k}{m} > \frac{b^2}{4m^2}$

  solution of this equation can be written

  $$x(t) = A\exp\left[-\frac{bt}{2m}\right]\cos(\omega_d t + \delta)$$

  where $\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
- Oscillations are approx. as in undamped case but amplitude decreases exponentially with time.



Damped SHM cont.

- Critical damping when $\frac{k}{m} = \frac{b^2}{4m^2}$
- Motion described by

$$x(t) = (C_1 + C_2 t)\exp\left[-\frac{b}{2m}t\right]$$

In this case no oscillations occur, system returns to equilibrium in minimum time. This is goal of designers of car suspension systems. Achieve by matching shock absorbers and springs in suspension system.

Damped SHM cont.

- If damping even stronger, we have $\frac{k}{m} < \frac{b^2}{4m^2}$
- Solution can be written in form

$$x(t) = \exp\left[-\frac{b}{2m}t\right]\left(C_1 \exp[\omega_h t] + C_2 \exp[-\omega_h t]\right)$$

where $\omega_h = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$

- Motion of system given impulse at x=0, t=0 is as below: