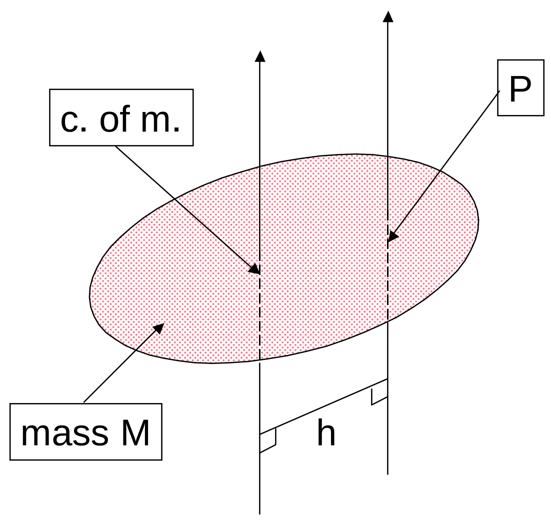
Lecture 8

- Parallel Axis Theorem
- Torque
- Newton's Second Law for Rotation
- Work, Power and Rotational Kinetic Energy
- More on Rotational Variables as Vectors
- More Torque

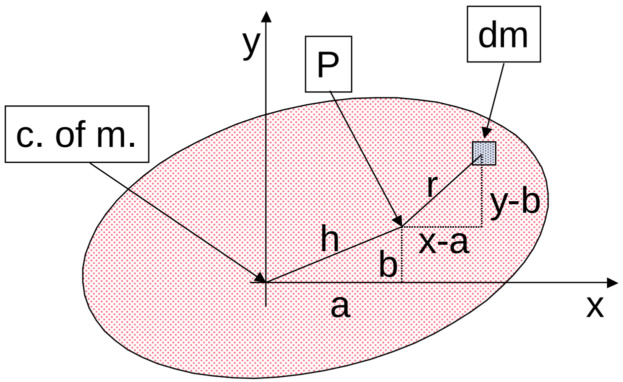
Parallel Axis Theorem

- Moment of inertia of body about axis through c. of m. is I_{cm}. Calc. moment of inertia about parallel axis.



Parallel axis theorem cont.

- View from above, place origin at c. of m., z axis along initial axis of rotation.



$$I_P = \int_M r^2 \, dm$$

$$= \int_M (x-a)^2 + (y-b)^2 \, dm$$

Parallel axis theorem cont.

$$I_P = \int_M x^2 + y^2 \, dm - 2a\int_M x \, dm -$$

$$2b\int_M y \, dm + \int_M a^2 + b^2 \, dm$$

Now 2nd and 3rd integrals zero from definition of c. of m. and choice of position of origin. First integral just definition of I_{cm}. *We also see:*

$$\int_M a^2 + b^2 \, dm = \int_M h^2 \, dm = Mh^2$$

Hence parallel axis theorem:

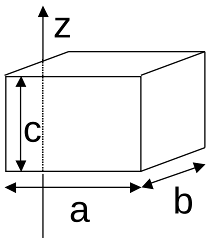
$$I_P = I_{cm} + Mh^2$$

Consequence, minimum moment of inertia for axes through c. of m.

Parallel Axis Theorem, an Example

- Consider again rectangular prism

$$I_e = \rho \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_0^a x^2 + y^2 \, dxdydz$$



$$= c\rho \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_0^a x^2 + y^2 \, dxdy$$

$$= c\rho \int_0^a x^2 y + \left. \frac{y^3}{12} \right|_{-\frac{b}{2}}^{\frac{b}{2}} dx$$

$$= bc\rho \left(\frac{x^3}{3} + \frac{b^2 x}{12} \right) \Bigg|_0^a = \frac{abc\rho}{12}(4a^2 + b^2)$$

$$= \frac{M}{12}(4a^2 + b^2)$$

Parallel Axis Theorem, an Example cont.

- Moment of inertia about c. of m.

$$I_{cm} = \frac{M}{12}(a^2 + b^2)$$

- Using parallel axis theorem

$$I_e = I_{cm} + M\left(\frac{a}{2}\right)^2$$

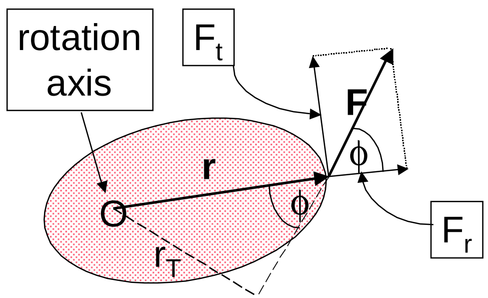
$$= \frac{M}{12}(a^2 + b^2) + \frac{M}{4} a^2$$

$$= \frac{M}{12}(4a^2 + b^2)$$

- Exercise, repeat check for axis along corner of prism and for other shapes.

Torque

- A force applied to a body may tend to rotate the body about an axis. Quantify using concept of torque (from latin “to twist”)
- Force in plane normal to axis

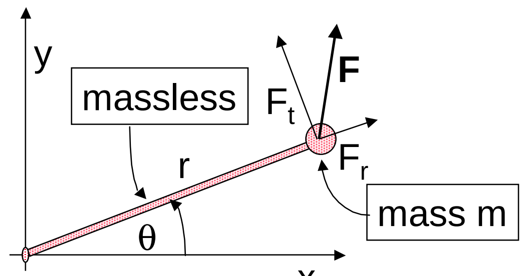


$$\tau = F_t r$$
$$= F r_T$$
$$= F r \sin\phi \quad \textit{units Nm}$$

Note $\vec{\tau} = \mathbf{r} \times \mathbf{F}$

Newton’s Second Law

- Consider rotation of a (simple) rigid body



- Relate tangential acceleration to torque

$$F_t = m a_t$$
$$\tau = F_t r$$
$$= m a_t r$$

Newton's second law cont.

- Consider rotational acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{1}{r}\frac{dv_t}{dt} = \frac{a_t}{r}$$

$$a_t = \alpha r$$

- Substitute for a_t

$$\tau = m(\alpha r)r = mr^2\alpha$$

- Recall definition of moment of inertia

$$I = mr^2$$

- Hence Newton's Second Law for rotation

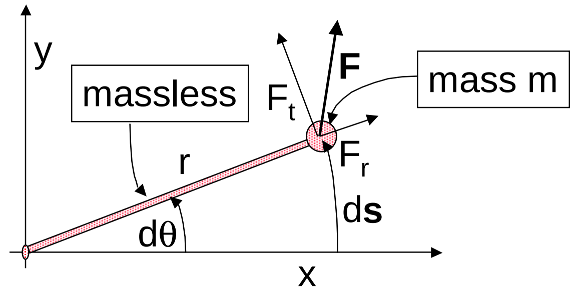
$$\tau = I\alpha$$

- If many forces applied

$$\sum \tau = I\alpha$$

Work and Rotation

- Consider same rigid body rotating through dθ under influence of force F



- Calculate work done

$$dW = \mathbf{F}\cdot d\mathbf{s} = F_t r\, d\theta = \tau\, d\theta$$

- For finite angular displacement

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta \qquad \textit{cf.}\ \ W = \int_{x_i}^{x^f} F\, dx$$

Power and Work K.E. Relation

- Consider power

$$P = \frac{d}{dt}W = \tau\frac{d\theta}{dt} = \tau\omega$$

- Now relate work to K.E. of rotation

$$W = \int_{\theta_i}^{\theta_f}\tau\, d\theta = \int_{\theta_i}^{\theta_f} I\alpha\, d\theta = \int_{\theta_i}^{\theta_f} I\frac{d\omega}{dt}d\theta$$

$$= \int_{\omega_i}^{\omega_f} I\frac{d\theta}{dt}d\omega = \int_{\omega_i}^{\omega_f} I\omega\, d\omega$$

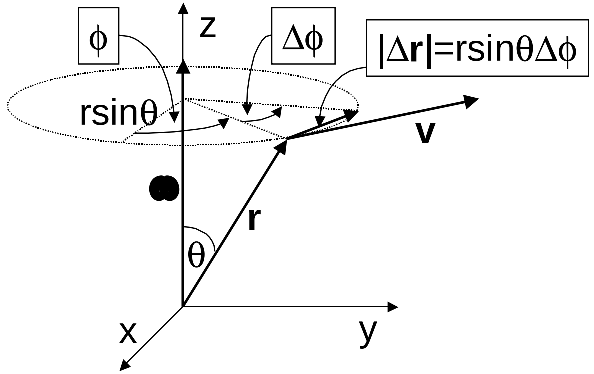
$$= \tfrac{1}{2}I\omega_f^{\,2} - \tfrac{1}{2}I\omega_i^{\,2}$$

$$= K_f - K_i$$

$$= \Delta K$$

More on Rotational Variables as Vectors

- Derive vector angular velocity



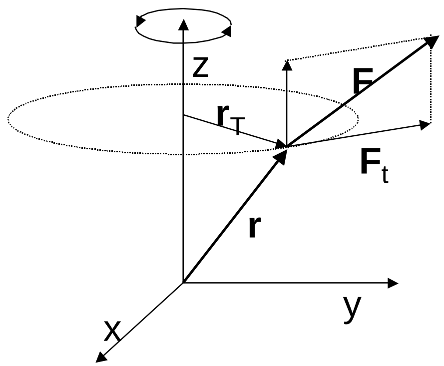
$$|\mathbf{v}| = \lim_{\Delta t \to 0}\frac{|\Delta\mathbf{r}|}{\Delta t} = \lim_{\Delta t \to 0}\frac{r\sin\theta\Delta\phi}{\Delta t}$$

$$= r\sin\theta\frac{d\phi}{dt} = r\omega\sin\theta = |\mathbf{r}\times\mathbf{v}|$$

As **v** *tangential to circle it is normal to plane containing* **r** *and* **ω**, *hence*
$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ *(using right hand rule)*

More Torque

- For force in plane normal to axis can write torque as $\tau = \mathbf{r} \times \mathbf{F}$
- Generalise for all forces.
- What is torque about given axis? Example, torque about z axis.



- Write $\mathbf{F}=(f_x,f_y,f_z)$ and $\mathbf{r}=(r_x,r_y,r_z)$ then $\mathbf{F}_t=(f_x,f_y,0)$ and $\mathbf{r}_T=(r_x,r_y,0)$.

More torque cont.

- From previous discussion we know torque about z is
  $\tau_z = \left|\mathbf{r}_T \times \mathbf{F}_t\right|$
- In terms of vector components this is

$$\tau_z = \left|(r_x, r_y, 0) \times (f_x, f_y, 0)\right|$$

$$= \begin{Vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & 0 \\ f_x & f_y & 0 \end{Vmatrix}$$

$$= r_x f_y - r_y f_x$$

More torque cont.

- Consider torque as vector and take component of vector along z axis to get torque about z

$$\vec{\tau} = \mathbf{r} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \left(r_y f_z - r_z f_y, r_z f_x - r_x f_z, r_x f_y - r_y f_x\right)$$

$$\tau_z = \vec{\tau} \cdot \hat{\mathbf{k}} = \begin{pmatrix} r_y f_z - r_z f_y \\ r_z f_x - r_x f_z \\ r_x f_y - r_y f_x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= r_x f_y - r_y f_x$$

- The correct result.