Lecture 5

- Force from potential
- Systems of particles
  - Centre of mass
  - Newton's second law for a system of particles
  - Linear momentum
  - Rocket equation
  - Kinetic energy

Force from Potential

- *Have seen*

  $U_g = mgy$ (gravity)

  $U_s = \frac{1}{2}kx^2$ (spring force)

- *Get force from potential*

  $$F_g = -\frac{d}{dy}U_g$$

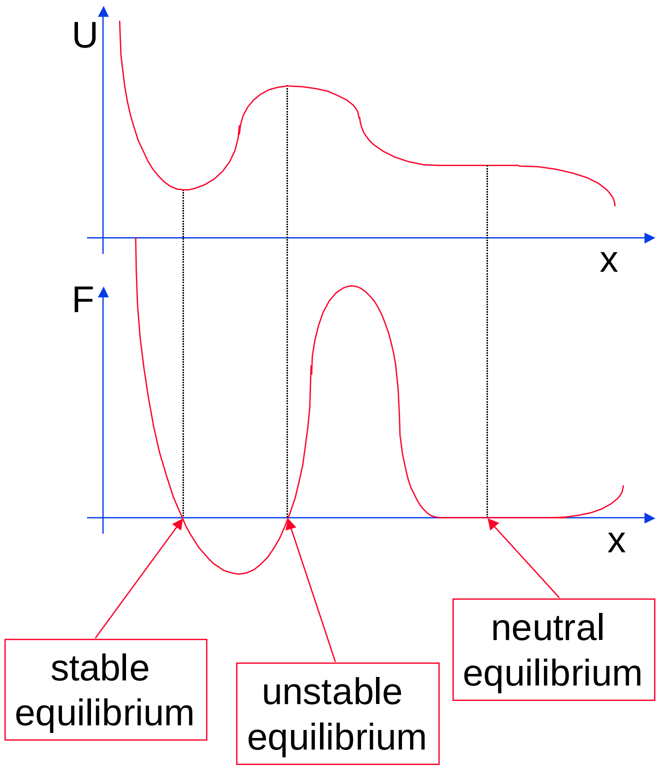
  $$F_s = -\frac{d}{dx}U_s$$

- *These are scalar equations, vector versions have form*

  $$\mathbf{F} = -\nabla U$$

  $$(F_x, F_y, F_z) = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$$

Force from potential cont.



Centre of mass

- The centre of mass of a system of particles is the point defined by:

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$$

$$M = \sum_i m_i$$

- We will show that the c. of m. moves like a point particle of mass M under the influence of the external forces acting on the system of particles.

Centre of mass cont.

- For continuous mass distributions where the density is given by $\rho(\mathbf{r})$ the c. of m. is defined by:

$$\mathbf{r}_{cm} = \frac{1}{M}\int_V \mathbf{r}dm = \frac{1}{M}\int_V \mathbf{r}\frac{dm}{dV}dV = \frac{1}{M}\int_V \mathbf{r}\rho(\mathbf{r})dV$$

- Consider rectangular prism with sides a, b and c, density ρ. Take origin at corner of cuboid.

$$M = abc\rho$$

C. of m. cont.

$$\mathbf{r}_{cm} = \frac{1}{M}\int_V \mathbf{r}\rho(\mathbf{r})dV$$

$$x_{cm} = \frac{1}{M}\int_0^c\int_0^b\int_0^a x\rho dxdydz = \frac{\rho}{M}\int_0^b\int_0^a xz\Big|_0^c dxdy = \frac{c\rho}{M}\int_0^b\int_0^a xdxdy = \frac{bc\rho}{M}\int_0^a xdx = \frac{a^2bc\rho}{2M} = \frac{a}{2}$$

Similarly $y_{cm}=b/2$ *and* $z_{cm}=c/2$

C. of m. cont.

- For uniform mass distributions with an axis or point of symmetry, the c. of m. is on that axis or at that point e.g.
  - Sphere, at centre
  - Cylinder, at midpoint of axis
  - Cuboid, at centre (see above)

Newton's Second Law for System of Particles

- Velocity of c. of m.

$$\mathbf{v}_{\mathbf{cm}} = \frac{d}{dt}\frac{1}{M}\sum_i m_i \mathbf{r}_\mathbf{i}$$

$$= \frac{1}{M}\sum_i m_i \frac{d}{dt}\mathbf{r}_\mathbf{i}$$

$$= \frac{1}{M}\sum_i m_i \mathbf{v}_\mathbf{i}$$

- Acceleration of c. of m.

$$\mathbf{a}_{\mathbf{cm}} = \frac{1}{M}\sum_i m_i \mathbf{a}_\mathbf{i}$$

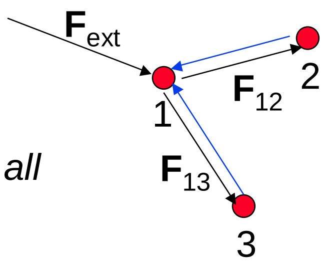
Newton's second law for system of particles cont.

- As individual particles obey Newton's laws

$$\sum \mathbf{F_i} = \sum m_i \mathbf{a_i}$$
$$= M\mathbf{a}_{cm}$$

Third law implies

$$\mathbf{F_{12}} = -\mathbf{F_{21}}$$
$$\mathbf{F}_{13} = -\mathbf{F_{31}}$$



Hence sum over all forces becomes

$$\sum \mathbf{F_i} = \sum \mathbf{F_{ext}}$$

For system of particles

$$\sum \mathbf{F_{ext}} = M\mathbf{a_{cm}}$$

Linear Momentum

- Total momentum of system of particles

$$\mathbf{P} = \sum m_i \mathbf{v_i}$$
$$= M\mathbf{v_{cm}}$$

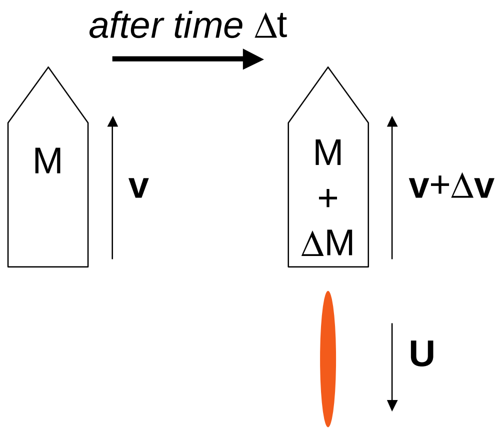
- Can write second law

$$\sum \mathbf{F_{ext}} = \frac{d}{dt}\mathbf{P}$$

- Conservation of linear momentum. If total external force zero, total momentum of system constant

Rocket Equation

- Consider motion of rocket plus expelled fuel.



Initial momentum is Mv
Momentum after Δt *is:*
(M+ ΔM)(v+ Δv)- ΔMU

Rocket equation cont.

- No external forces (no gravity!) so equate momenta

$$Mv = (M + \Delta M)(v + \Delta v) - \Delta MU$$
$$\approx Mv + M\Delta v + \Delta M(v - U)$$
$$M\Delta v = \Delta M(U - v)$$

Divide by Δt *and take limit of vanishing* Δt *.*

$$M\frac{d}{dt}v = (U - v)\frac{d}{dt}M$$

Use fuel speed w.r.t rocket, u

$$u = v - U$$

$$M\frac{d}{dt}v = -u\frac{d}{dt}M$$

Rocket equation cont.

- Define rate at which rocket uses fuel $R = -dM/dt$
- Get first rocket equation Ru=ma
- Ru termed thrust of rocket
- Calc. final vel. v_f of rocket starting with vel. v_i mass m_i and finishing with mass m_f

$$dv = -u\frac{dM}{M}$$

$$\int_{v_i}^{v_f} dv = -u\int_{m_i}^{m_f}\frac{dM}{M}$$

$$v_f - v_i = -u(\ln m_f - \ln m_i)$$

$$= u\ln\left(\frac{m_i}{m_f}\right) \quad \textit{second rocket eqn.}$$

Kinetic Energy

- Distinguish between translational K.E. K_{cm}, due to motion of centre of mass, and K.E. due to motion w.r.t. c. of m. The latter, plus P.E. due to mutual interactions of particles in body, contribute to internal energy.

$$K_{cm} = \tfrac{1}{2}M\mathbf{v}_{cm}^{\;2}$$

- Change in K.E. due to external force on body is work done by external force in moving c. of m.

$$\Delta K_{cm} = \mathbf{F}_{\mathbf{ext}} \cdot \mathbf{s}_{\mathbf{cm}}$$

$$= W_{cm}$$

Kinetic energy cont.

- W_{cm} not necessarily actual work done by $\mathbf{F}_{ext}$, force may also cause changes in internal energy.
- Defining W_{ext} to be actual work done by external forces:

$$\Delta K_{cm} + \Delta U + \Delta E_{int} = W_{ext}$$