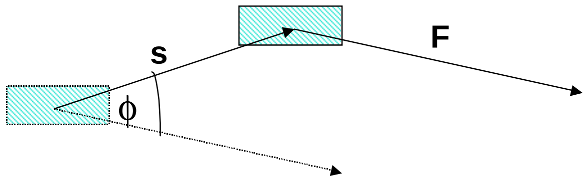
Lecture 3

- Work
  - Definition
  - Variable forces, 1D
  - Variable forces, 3D
- Energy
  - Kinetic energy
  - Work and kinetic energy
- Power

Work

- The work done by a constant force **F** in moving a particle through a displacement **s** is defined to be:

$$W = \mathbf{F} \cdot \mathbf{s}$$
$$= Fs \cos \phi$$

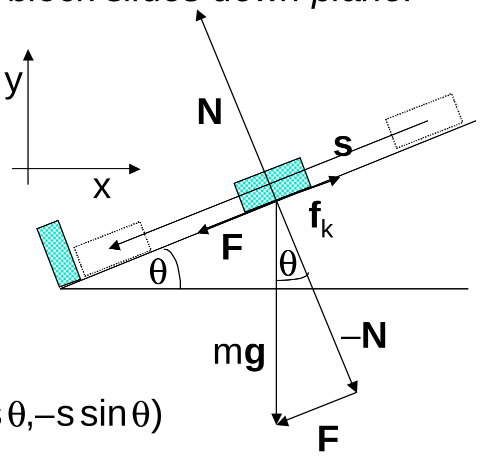


- SI unit of work is the Joule (J),

  1 J = 1 Nm

Work cont.

Consider work done by gravity and friction when block slides down plane.



$$W_g = m\mathbf{g}\cdot\mathbf{s}$$

$$\mathbf{g} = (0,-g)$$

$$\mathbf{s} = (-s\cos\theta, -s\sin\theta)$$

$$W_g = mgs\sin\theta$$

$$W_f = \mathbf{f}_k\cdot\mathbf{s}$$

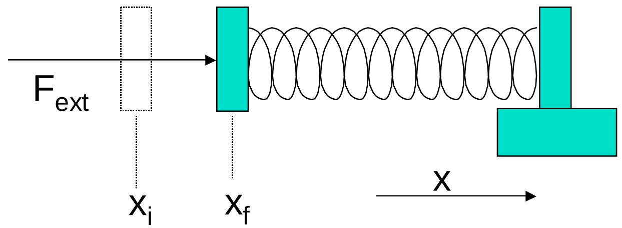
$$\mathbf{f_k} = \mu_k mg\cos\theta(\cos\theta, \sin\theta)$$

$$W_f = \mu_k mgs\cos\theta(-\cos^2\theta - \sin^2\theta)$$

$$= -\mu_k mgs\cos\theta$$

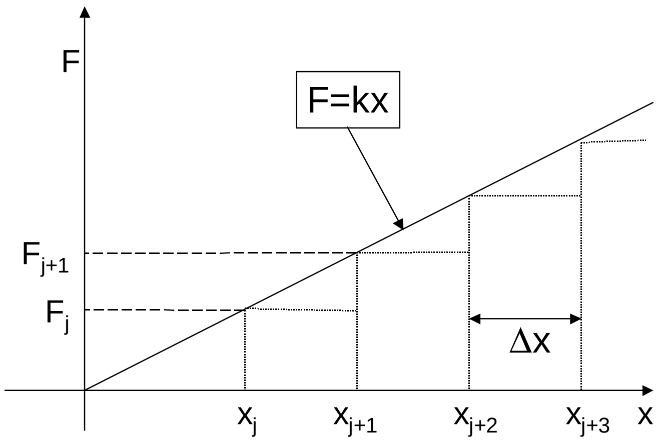
Work and varying forces, 1D.

- Split particle path into elements along which force approx. const.
- Add up work for each element.
- Limit of large no. of elements gives work for varying force.
- Example, compressing a spring.



- Hooke's law, F_s=-kx, where k is spring constant.
- Newton's third law, F_{ext}=-F_s

Compressing a spring cont.



$$W \approx \sum_j F_j \Delta x$$

$$= \int_{x_i}^{x_f} F dx$$

Compressing a spring cont.

Work done by force:

$$W = k\int_{x_i}^{x_f} x dx$$

$$= \tfrac{1}{2}k(x_f^2 - x_i^2)$$

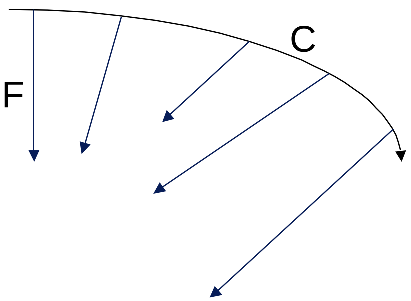
If define $x_i=0$ *then:*

$$W = \tfrac{1}{2}kx_f^2$$

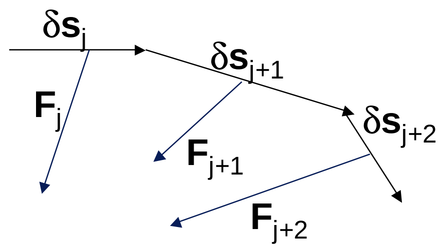
The spring will do this amount of work as it expands.
The spring has gained potential energy $U = \tfrac{1}{2}kx_f^2$

Work and varying forces, more than 1D.

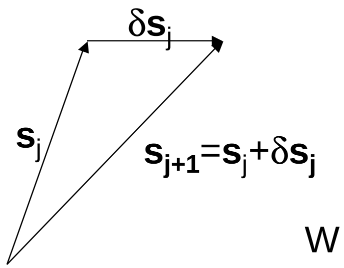
- Split particle path into elements along which force approx. const.
- Add up work for each element.
- Limit of large no. of elements gives work for varying force



Work and varying forces cont.



Where $\delta \mathbf{s}_j = \mathbf{s}_{j+1} - \mathbf{s}_j$

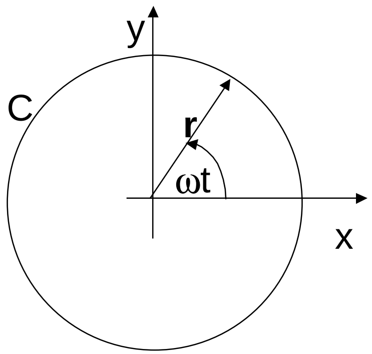


$$W \approx \sum_j \mathbf{F}_j \cdot \delta \mathbf{s}_j$$

$$= \int_C \mathbf{F} \cdot d\mathbf{s}$$

Work and varying forces cont.

Example 1
Calculate work done by centripetal force $\mathbf{F}$=-m$\varpi^2\mathbf{r}$ during turn round circle.



$$\mathbf{r} = r(\cos\omega t, \sin\omega t)$$

$$\mathbf{F} = -mr\omega^2(\cos\omega t, \sin\omega t)$$

Work and varying forces cont.

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt$$

$$= \int_0^{\frac{2\pi}{\omega}} F_x \frac{dr_x}{dt}dt + \int_0^{\frac{2\pi}{\omega}} F_y \frac{dr_y}{dt}dt$$

$$= \int_0^{\frac{2\pi}{\omega}} -mr\omega^2\cos\omega t(-r\omega\sin\omega t)\,dt +$$

$$\int_0^{\frac{2\pi}{\omega}} -mr\omega^2\sin\omega t(r\omega\cos\omega t)\,dt$$

$$= 0$$

Work and varying forces cont.

Example 2

Find the work done by the force **F**=3xy**i**-2yz**j**+y**k** (in N) moving a particle along the curve given by **s**=t**i**+2t^2**j**+(1+t)**k** from t=1 to t=2 (distances in m).

$$W = \int_C \mathbf{F} \cdot d\mathbf{s}$$

$$= \int_C (3xy\,\mathbf{i} - 2yz\,\mathbf{j} + y\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3xy\,dx - 2yz\,dy + y\,dz)$$

Work and varying forces cont.

$$W = \int_1^2 [3(t)(2t^2)dt -$$

$$2(2t^2)(1+t)d(2t^2) + 2t^2 d(1+t)]$$

$$= \int_1^2 (6t^3 - 16t^3 - 16t^4 + 2t^2)dt$$

$$= -\frac{16t^5}{5} - \frac{5t^4}{2} + \frac{2t^3}{3}\Bigg|_1^2$$

$$= -\frac{3961}{30} \text{ J}$$

Kinetic Energy

Newton's Second Law, work and kinetic energy.

$$\mathbf{F} = \frac{d}{dt}\mathbf{p} = m\frac{d}{dt}\mathbf{v}$$

$$\mathbf{F}\cdot d\mathbf{r} = m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r}$$

$$= m\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{r}}{dt}dt = m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}dt$$

Now $\frac{d}{dt}\mathbf{v}^2 = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}$

$$\Rightarrow \frac{d}{dt}\frac{m\mathbf{v}^2}{2} = m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}$$

Hence $\mathbf{F}\cdot d\mathbf{r} = \frac{d}{dt}\left(\frac{m\mathbf{v}^2}{2}\right)dt$ and

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_i}^{t_f}\frac{d}{dt}\frac{m\mathbf{v}^2}{2}dt = \frac{1}{2}\left(mv_f^2 - mv_i^2\right)$$

Kinetic energy cont.

- Define kinetic energy

$$K = \frac{1}{2}m\mathbf{v}^2$$

- Work done on particle equal to gain in kinetic energy.

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K$$

Power

- Power is the rate of work.

$$P = \frac{d}{dt} W$$

- Instantaneous power

$$P = \frac{d}{dt}\mathbf{F}\cdot\mathbf{r} = \mathbf{F}\cdot\frac{d}{dt}\mathbf{r} = \mathbf{F}\cdot\mathbf{v}$$

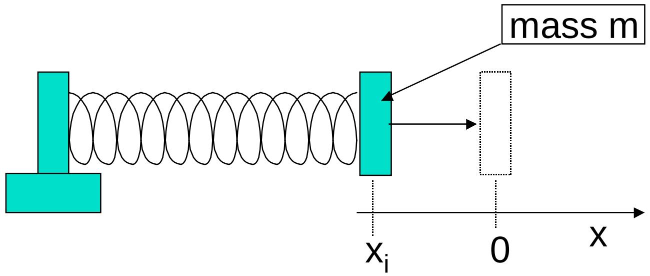
- Average power

$$\overline{P} = \frac{W}{\Delta t}$$

- SI unit of power is the Watt (W)

  1 W = 1 J/s

Power released when spring expands



- Instantaneous power

$$P = \frac{d}{dt} W = F\frac{dx}{dt} = Fv = ?$$

- Average power

$$\overline{P} = \frac{kx_i^2}{2\Delta t} = ?$$

- We shall return to this problem!