Partial differential equations

- In this lecture we will:
  - Introduce a classification scheme for partial differential equations (PDEs).
  - Revisit the superposition theorem.
  - Derive the partial differential equation that describes the wave motion of an elastic string.
  - Solve the PDE by separating variables.
- A comprehension question for this lecture:
  - What is the order of the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}?$$

  - Is this equation linear?
  - Is it homogeneous?

Classifying PDEs

- PDE classification is similar to that for ordinary differential equations (ODEs).
- The order is given by the highest derivative, e.g. the 1D heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

 is second order.
- The equation is linear if the dependent variable (u) and its derivatives appear only to the first power (the heat equation is linear).
- The equation is homogeneous if every term contains the dependent variable or one of its derivatives (the heat equation is homogeneous).

Principle of superposition

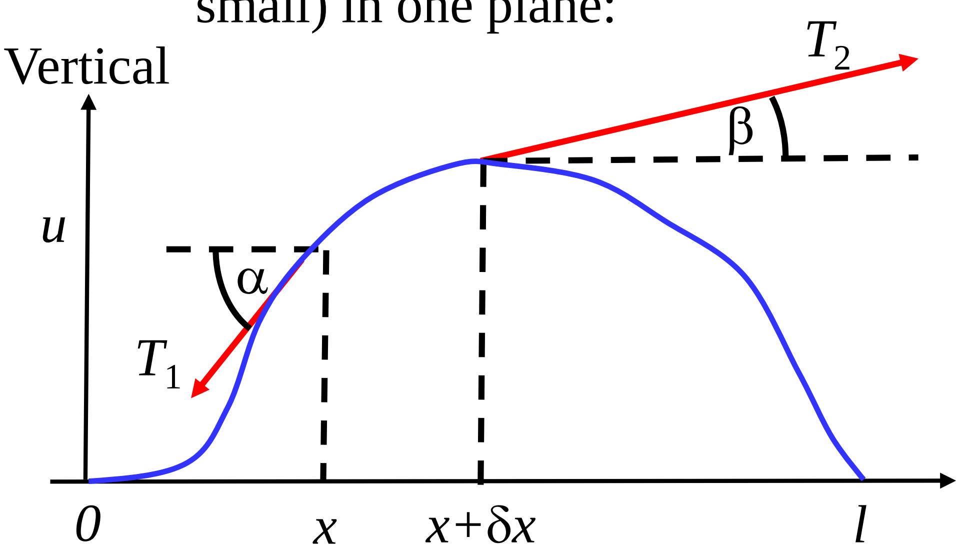
- Another similarity to ODEs!
- If u_1 and u_2 are solutions of a linear PDE, then:

$$u = c_1 u_1 + c_2 u_2,$$

 where c_1 and c_2 are constants, is also a solution of the PDE.
- The proof of this is similar to the proof for the ODE case…
- …and is left as an exercise for the student!

Equation of motion of string

- Want to work out how string behaves, assume:
  - Homogeneous, with mass per unit length ρ.
  - Tension much larger than gravity.
  - Small motions (i.e. α and β small) in one plane:



- No motion in horizontal direction:

$$T_1 \cos\alpha = T_2 \cos\beta \approx T.$$

- Vertical motion, Newton's second law gives:

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\,\delta x \frac{\partial^2 u}{\partial t^2}.$$

- Using first equation:

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho}{T}\delta x \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\tan\beta - \tan\alpha}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

- Now:

$$\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_x \text{ and } \tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x+\delta x}.$$

Equation of motion of string

- So we have:

$$\frac{\tan\beta - \tan\alpha}{\delta x} = \frac{1}{\delta x}\left(\left.\frac{\partial u}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial u}{\partial x}\right|_{x} \right),$$

- and hence:

$$\frac{1}{\delta x}\left(\left.\frac{\partial u}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial u}{\partial x}\right|_{x} \right) = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

- Letting dx $\rightarrow$ 0 gives:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

- This is the 1D wave equation, generally written:

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}.$$

- (Use c^2 to indicate constant positive!)

- Solution of equation is function $u(x, t)$.
- Have boundary conditions $u(0, t) = 0$ and $u(l, t) = 0$ (string fixed at ends).
- At $t = 0$, initial deflection is $f(x)$ and initial velocity is $g(x)$.
- This means:

$$u(x,0) = f(x) \text{ and } \frac{\partial u(x,0)}{\partial t} = g(x).$$

- Need solution that satisfies these conditions!
- Three steps:
  - Separate variables, get 2 ODEs.
  - Solve ODEs satisfying boundary conditions.
  - Put these solutions together to solve PDE.

Solving equation of motion of string – step one

- Assume can write solution in form:

$$u(x,t) = F(x)G(t).$$

- Differentiating gives:

$$\frac{\partial u}{\partial x} = F'G, \quad \frac{\partial^2 u}{\partial x^2} = F''G \text{ and}$$

$$\frac{\partial u}{\partial t} = F\dot{G}, \quad \frac{\partial^2 u}{\partial t^2} = F\ddot{G}.$$

- Our wave equation becomes:

$$F''G = c^2 F\ddot{G}.$$

- Rearranging:

$$\frac{F''}{F} = \frac{c^2\ddot{G}}{G}.$$

- LHS depends only on *x*, RHS on *t*, so must both be equal to a constant, *k*.

- We have:

$$\frac{F''}{F} = k, \quad \frac{c^2\ddot{G}}{G} = k.$$

- This gives:

$$F'' - kF = 0$$

- and

$$\ddot{G} - c^2 kG = 0.$$

- These are two ODEs that we can solve using the techniques we have already developed…
- …while ensuring that the boundary conditions are satisfied, i.e. we need:

$$F(0) = 0 \text{ and } F(l) = 0.$$

Solving equation of motion of string – step two

- First look at positive $k = \mu^2$:

$$F'' - \mu^2 F = 0.$$

- Hence:

$$F = Ae^{\mu x} + Be^{-\mu x}.$$

- But $F(0) = 0$ and $F(l) = 0$ force $A = 0$ and $B = 0$, so $F = 0$: not useful!
- Try negative $k = -p^2$:

$$F'' + p^2 F = 0.$$

- This gives:

$$F = A\cos px + B\sin px.$$

- The boundary conditions then give:

$$F(0) = A = 0 \text{ and } F(l) = B\sin pl = 0.$$

- This means:

$$pl = n\pi \text{ or } p = \frac{n\pi}{l}.$$

- Setting $B = 1$, we have an infinite number of solutions of the form:

$$F_n(x) = \sin\frac{n\pi}{l}x.$$

- The equation for G with $k = -(n\pi/l)^2$ is:

$$\ddot{G} + c^2\left(\frac{n\pi}{l}\right)^2 G = 0.$$

- Writing $\lambda_n = cn\pi/l$, we get:

$$\ddot{G} + \lambda_n^2 G = 0.$$

- This has solutions:

$$G_n(t) = A_n\cos\lambda_n t + B_n\sin\lambda_n t.$$

- Hence a solution of the PDE is:

$$u_n(x,t) = \left(A_n\cos\lambda_n t + B_n\sin\lambda_n t\right)\sin\frac{n\pi}{l}x.$$

Solving equation of motion of string – step three

- Some jargon:
- The $u_n(x, t)$ are called eigenfunctions and the λ_n eigenvalues (or characteristic functions and values, respectively).
- The eigenvalue set λ_1, λ_2, λ_3… is called the spectrum.
- The motion with of the string with wavelength λ_n is called the n^{th} normal mode.
- In order to satisfy the initial conditions (the shape and velocity of string at $t = 0$), we need to exploit the superposition theorem...

- …write the solution in the form:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(A_n \cos\lambda_n t + B_n \sin\lambda_n t\right) \sin\frac{n\pi}{l} x.$$

- Then $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l} x = f(x)$ and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(-A_n\lambda_n \sin\lambda_n t + B_n\lambda_n \cos\lambda_n t\right) \sin\frac{n\pi}{l} x\Bigg|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n\lambda_n \sin\frac{n\pi}{l} x = g(x).$$

- Choosing the A_n to be the Fourier coefficients for $f(x)$ and the B_n to be those for $g(x)$ ensures that the initial conditions are satisfied.

An example – initial deflection triangle

- Find solution to 1D wave equation with initial conditions $g(x) = 0$ and

$$f(x) = \begin{cases} \frac{2k}{l} x \text{ if } 0 < x < \frac{l}{2}, \\ \frac{2l}{l}(l - x) \text{ if } \frac{l}{2} < x < l. \end{cases}$$

- $g(x) = 0$ implies $B_n = 0$ for all n.
- Fourier analysis of $f(x)$ gives:

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{l^2}\sin\frac{\pi}{l}x - \frac{1}{3^2}\sin\frac{3\pi}{l}x + \ldots\right)$$

- Hence:

$$u(x,t) = \frac{8k}{\pi^2}\left(\frac{1}{l^2}\sin\frac{\pi}{l}x\cos\frac{\pi c}{l}t - \frac{1}{3^2}\sin\frac{3\pi}{l}x\cos\frac{3\pi c}{l}t + \ldots\right)$$