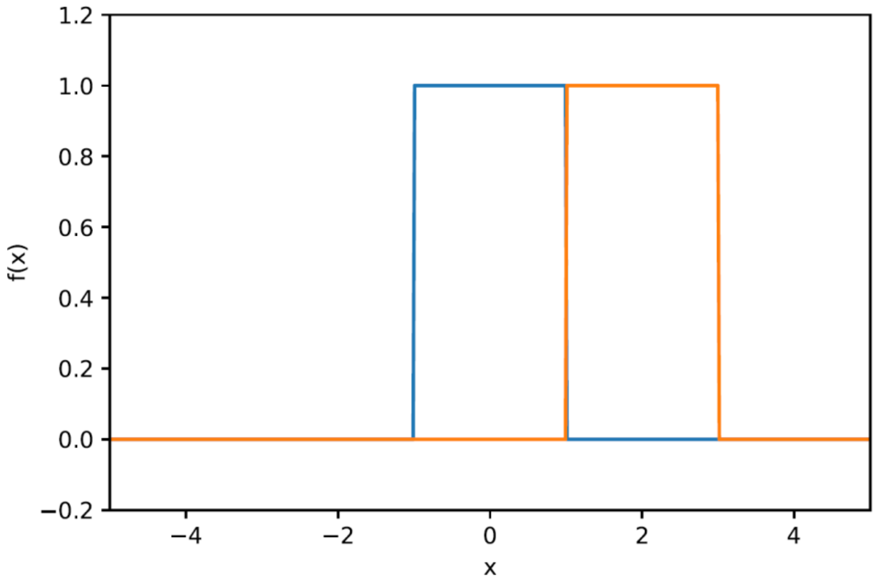
# Fourier transforms

- In this lecture we will:
  - Look at some more Fourier transforms.
  - See how Fourier transforms can be used to solve differential equations.
  - Do a useful integral.
- A comprehension question for this lecture:
  - Calculate the Fourier transform of the function given by:

$$f(x) = 1 \text{ if } -2 < x < 0,$$
$$= 0 \text{ otherwise.}$$

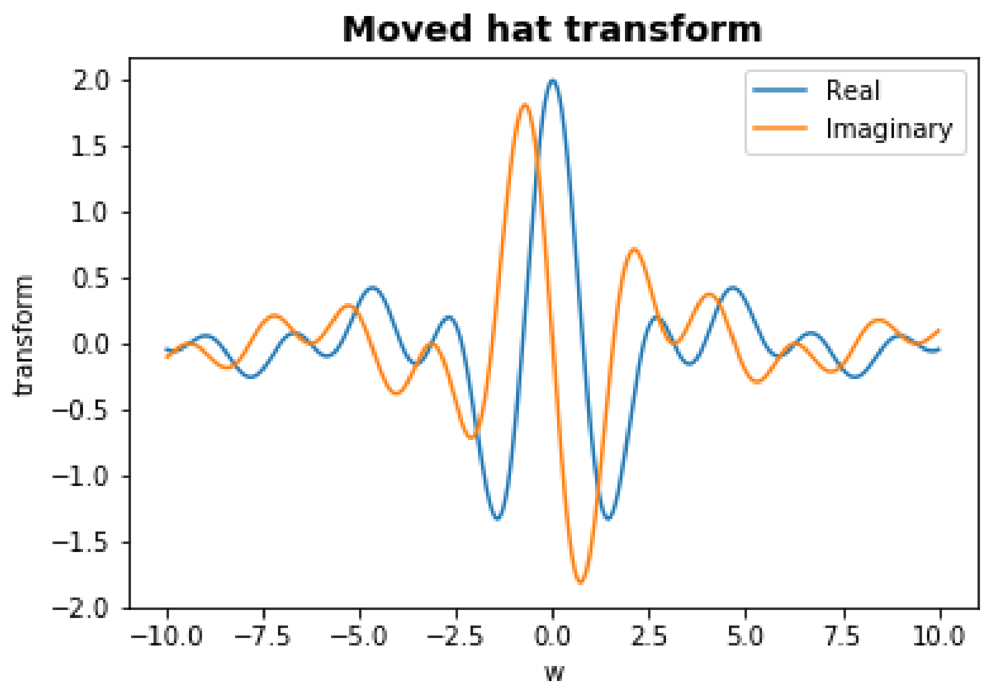
# Fourier transforms – effect of shifting function

- Shift hat from origin.




- Is this function (the orange one!) even or odd?
- Would you expect the transform to be purely real ("cosine")…
- …or purely imaginary ("sine")?

- Fourier transform of shifted hat:



# Shifted hat transform

- Fourier transform of hat is:

$$\tilde{f}(\omega) = 2\frac{\sin\omega}{\omega}.$$

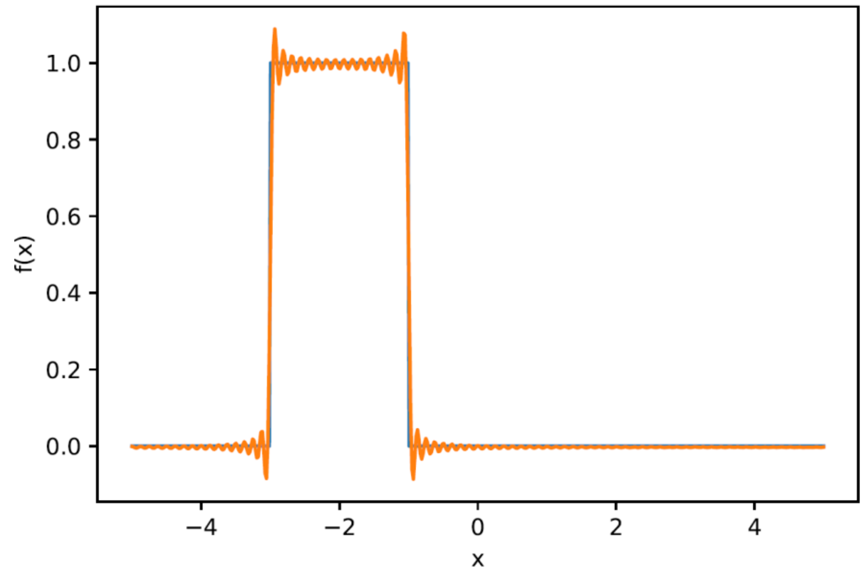
- Fourier transform of hat shifted to right is:

$$\tilde{f}(\omega) = \exp[-2i\omega]\times\frac{2\sin\omega}{\omega}.$$

- Given this, what function would you expect the following transform to represent:

$$\tilde{f}(\omega) = \exp[2i\omega]\times\frac{2\sin\omega}{\omega}?$$

- Hat shifted to left!



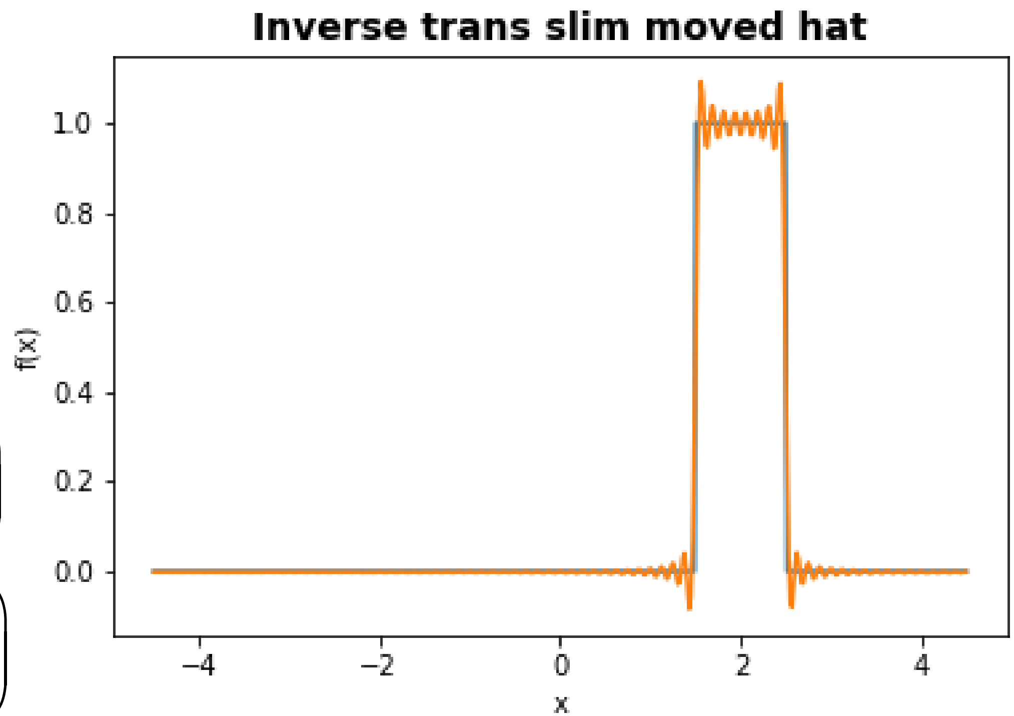
# Slim hat

- Yet another function.

$$f(x) = 1 \text{ if } 3/2 < x < 5/2$$
$$= 0 \text{ otherwise.}$$

- Fourier transform:

$$\tilde{f}(\omega) = \int_{3/2}^{5/2} \exp[-i\omega x]\,dx$$
$$= -\frac{1}{i\omega}\left(\exp\left[-\frac{5i\omega}{2}\right] - \exp\left[-\frac{3i\omega}{2}\right]\right)$$
$$= \frac{2}{\omega}\frac{1}{2i}\left(\exp\left[-\frac{3i\omega}{2}\right] - \exp\left[-\frac{5i\omega}{2}\right]\right)$$
$$= \frac{2}{\omega}\frac{1}{2i}\exp\left[-\frac{4i\omega}{2}\right]\left(\exp\left[\frac{i\omega}{2}\right] - \exp\left[-\frac{i\omega}{2}\right]\right)$$
$$= \frac{2}{\omega}\exp[-2i\omega]\sin\left(\frac{\omega}{2}\right).$$



# Transform of exponential

- E.g. $(a > 0)$:

$$f(x) = \begin{cases} \exp[-ax] \text{ if } x \geq 0 \\ 0 \text{ otherwise,} \end{cases}$$

- Fourier transform is:

$$\tilde{f}(\omega) = \int_0^\infty \exp[-ax]\exp[-i\omega x]dx$$

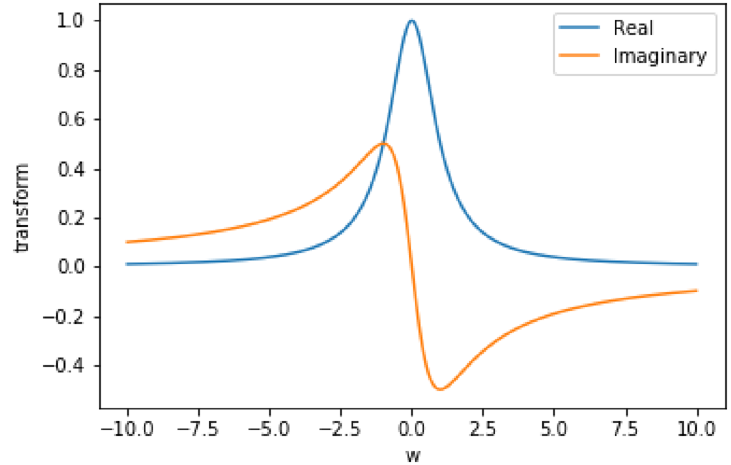
$$= \int_0^\infty \exp[-(a+i\omega)x]dx$$

$$= -\left[\frac{\exp[-(a+i\omega)x]}{a+i\omega}\right]_0^\infty$$

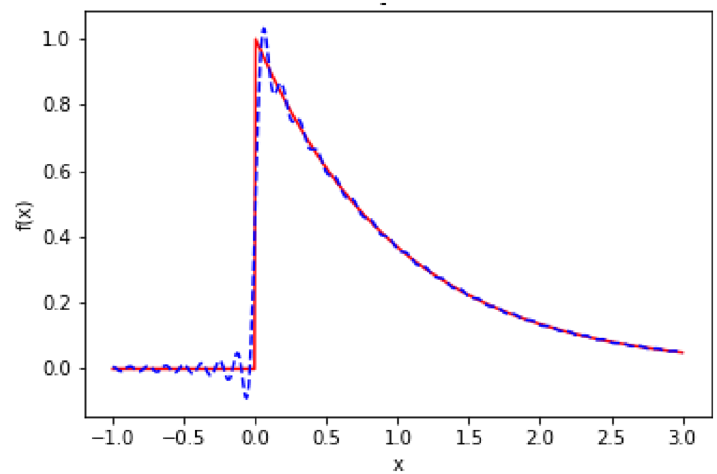
$$= -\left[\frac{\exp[-ax]\exp[-i\omega x]}{a+i\omega}\right]_0^\infty = \frac{1}{a+i\omega}.$$

- Have used: $\exp[-ax] \to 0$ as $x \to \infty$.

- Transform:




- Recovered function:



# Fourier transform of derivative

- One of the useful properties of Fourier transforms derives from the following result (for functions such that $f(x) \to 0$ as $x \to \pm\infty$):

$$\tilde{f}'(\omega) = \int_{-\infty}^{\infty} f'(x)\exp(-i\omega x)\,dx$$
$$= \left[f(x)\exp(-i\omega x)\right]_{-\infty}^{\infty}$$
$$-\int_{-\infty}^{\infty} f(x)(-i\omega)\exp(-i\omega x)\,dx$$
$$= i\omega\int_{-\infty}^{\infty} f(x)\exp(-i\omega x)\,dx$$
$$= i\omega\tilde{f}(x).$$

- Applying this twice gives:

$$\tilde{f}''(\omega) = -\omega^2\tilde{f}(x).$$

# Differential equations and Fourier transforms

- This can be used to solve some differential equations, for example:

$$ay''(x) + by'(x) + cy(x) = f(x).$$

- Take the Fourier transform of both sides:

$$a\tilde{y}''(\omega) + b\tilde{y}'(\omega) + c\tilde{y}(\omega) = \tilde{f}(\omega).$$

Hence:

$$a(-\omega^2)\tilde{y}(\omega) + b(i\omega)\tilde{y}(\omega) + c\tilde{y}(\omega) = \tilde{f}(\omega)$$
$$\Rightarrow (-a\omega^2 + i\omega b + c)\tilde{y}(\omega) = \tilde{f}(\omega)$$
$$\Rightarrow \tilde{y}(\omega) = \frac{\tilde{f}(\omega)}{-a\omega^2 + i\omega b + c}.$$

- Then use inverse Fourier transform to determine y(x).

# One more useful Fourier transform

- Look at $f(x) = \exp[-a^2 x^2]$.
- Calculate the transform:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \exp[-a^2 x^2]\exp[-i\omega x]\,dx$$

$$= \int_{-\infty}^{\infty} \exp[-(a^2 x^2 + i\omega x)]\,dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-a^2\left[\left(x + \frac{i\omega}{2a^2}\right)^2 + \frac{\omega^2}{4a^4}\right]\right)dx$$

$$= \int_{-\infty}^{\infty} \exp\left[-\frac{\omega^2}{4a^2}\right]\exp\left(-a^2\left[x + \frac{i\omega}{2a^2}\right]^2\right)dx$$

$$= \exp\left[-\frac{\omega^2}{4a^2}\right]\int_{-\infty}^{\infty} \exp[-a^2 y^2]\,dy.$$

- Using the result (see next slide):

$$\int_{-\infty}^{\infty} \exp[-a^2 y^2]\,dy = \frac{\sqrt{\pi}}{a}.$$

- We have:

$$\tilde{f}(\omega) = \frac{\sqrt{\pi}}{a}\exp\left[-\frac{\omega^2}{4a^2}\right].$$

- In this case, the Fourier transform has the same functional form (exponential) as the function.

# A useful integral

- Define:

$$I = \int_{-\infty}^{\infty} \exp[-a^2 x^2]\, dx.$$

- Then:

$$I^2 = \left( \int_{-\infty}^{\infty} \exp[-a^2 x^2]\, dx \right)^2$$

$$= \int_{-\infty}^{\infty} \exp[-a^2 x^2]\, dx \int_{-\infty}^{\infty} \exp[-a^2 y^2]\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-a^2 x^2] \exp[-a^2 y^2]\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-a^2 (x^2 + y^2)]\, dx\, dy$$

- Think of this as an integral over the (x, y) plane and convert to polar coordinates:

$$x = r \cos\theta,\ y = r \sin\theta$$

$$dx\, dy = r\, dr\, d\theta.$$

- We then have:

$$I^2 = \int_0^{\infty} \int_0^{2\pi} \exp[-a^2 r^2]\, r\, dr\, d\theta$$

$$= \int_0^{\infty} \exp[-a^2 r^2]\, r\, dr \int_0^{2\pi} d\theta.$$

- Using:

$$s = r^2 \Rightarrow ds = 2r\, dr \Rightarrow dr = \frac{ds}{2r}$$ we get:

- Hence:

$$I^2 = \int_0^{\infty} \exp[-a^2 s]\, r \frac{ds}{2r} [\theta]_0^{2\pi}$$

$$= \frac{1}{-2a^2} \exp[-a^2 s] \Big|_0^{\infty} \times 2\pi$$

$$= \left( 0 - \frac{-1}{2a^2} \right) \times 2\pi = \frac{\pi}{a^2}.$$

- This gives: $I = \sqrt{\pi}/a$.

# Fourier series and transforms in physics

- The time development of many physical systems is described by partial differential equations (involving say position x and time t).
- Often we know the initial configuration, e.g. as a function of x at time zero.
- In the case of a periodic initial configuration, or one that is confined to a finite region of x, it is often useful to write this configuration as a Fourier series.
- Each mode in the series typically has simple (but different) behaviour as a function of t.
- Each mode can therefore be solved and its behaviour with t calculated.
- The behaviour of the system can then be found by summing up the solutions for the individual modes.
- Examples include heat diffusing along a metal bar or waves on strings.
- A similar procedure can be used for non-periodic configurations and those not confined to a limited x range.
- In these cases, Fourier transforms are used rather than Fourier series.
- Examples include waves travelling through space and single pulses in electronic circuits.