Fourier transforms

- In this lecture we will:
  - See how Fourier series can be written in exponential form.
  - Introduce Fourier transforms.
  - Look at some examples to try and gain a little insight into the Fourier transform.

- A comprehension question for this lecture:
  - Show that the Fourier transform of the function:

    $$f(x) = -1 \text{ if } -1 < x < 1, \quad = 0 \text{ otherwise,}$$

    is $\tilde{f}(\omega) = -\frac{2}{\omega}\sin\omega.$

Fourier series using exponentials

- The "standard" formulae for finding Fourier coefficients are:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos\frac{2n\pi t}{T}\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\sin\frac{2n\pi t}{T}\,dt$$

- Now $\exp[i\theta] = \cos\theta + i\sin\theta$...

- …so could re-write the formulae as:

$$w_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\left(\exp\left[-\frac{2in\pi t}{T}\right]\right)dt \text{ for } n \neq 0$$

$$w_0 = a_0$$

- These contain the same information as the standard formulation:

$$w_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\left(\cos\left[-\frac{2n\pi t}{T}\right] + i\sin\left[-\frac{2n\pi t}{T}\right]\right)dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} g(t)\left(\cos\left[\frac{2n\pi t}{T}\right] - i\sin\left[\frac{2n\pi t}{T}\right]\right)dt$$

$$= a_n - ib_n$$

Fourier series using exponentials

- Using this formulation, the Fourier series representation of the function becomes:

$$g(t) = a_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[w_n \exp\left(\frac{2in\pi t}{T} \right) \right].$$

- This gives us the required result because:

$$\begin{aligned} g(t) &= a_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[\left(a_n - ib_n\right)\left(\cos\frac{2n\pi t}{T} + i\sin\frac{2n\pi t}{T} \right) \right] \\ &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T} \right). \end{aligned}$$

- We can go one step further…

Fourier series using exponentials

- Allow negative values of n.
- We then see that, because cosine is even and sine odd, we get coefficients such that:

  $$a_n = a_{-n}$$

  $$b_n = -b_{-n}$$

- We can then write our function as:

  $$g(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{Re}\left[w_n \exp\left(\frac{2in\pi t}{T}\right)\right].$$

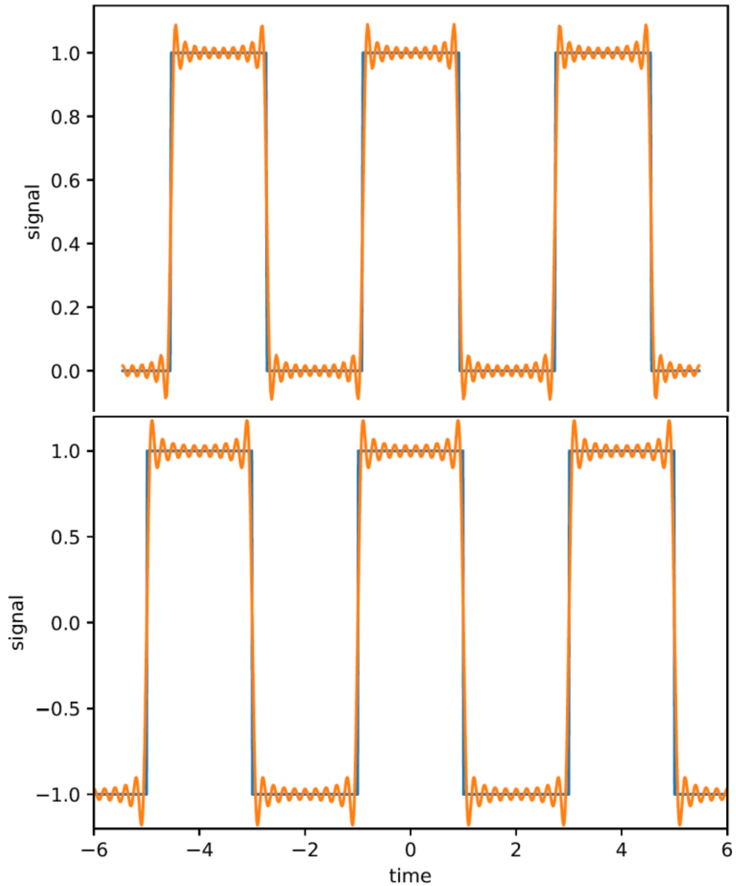
- The factor of ½ is needed as all terms appear twice (once with negative and once with positive n), except for the case where n = 0.
- This also allows us to use the same formula to determine w_0 as all the other coefficients, so:

  $$w_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\left(\exp\left[-\frac{2in\pi t}{T}\right]\right)dt.$$

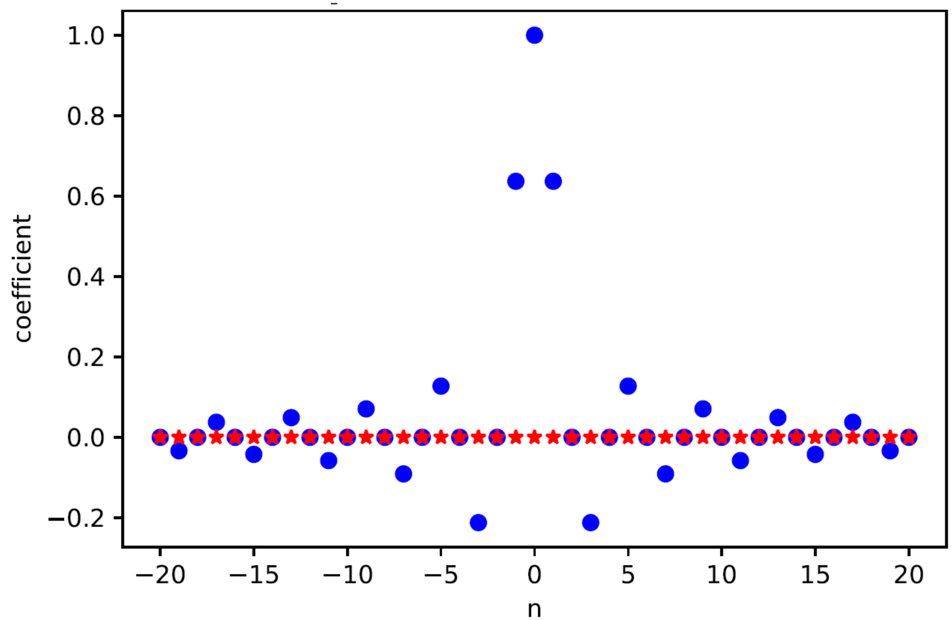
- Note that $w_0 = 2a_0$.
- (We may have to fix the w_0 calc. by hand if n appears in the denominator, as in the case of the square wave.)
- This will make it easier to see the relationship between the Fourier series and the Fourier transform.
- First, check it all works for the square wave (with “fix” for w_0!).

Fourier series using exponentials

- Square wave, top using exponential, bottom using standard Fourier series (both with 20 terms).



- Plot the coeffs w_n as a function of n.
- The real part (a_n) is shown as blue dots and the imaginary part (b_n) as red stars:



- The b_n are all zero, as correspond to sine (odd) terms and function is even.

Fourier series and transforms

- Fourier series can describe periodic functions (e.g. square wave).
- If need to describe single pulse (e.g. “top hat”), need to move from using cosines and sines with frequencies 0, f, 2f, 3f… to using full frequency spectrum.
- Make f continuous, so (schematically):

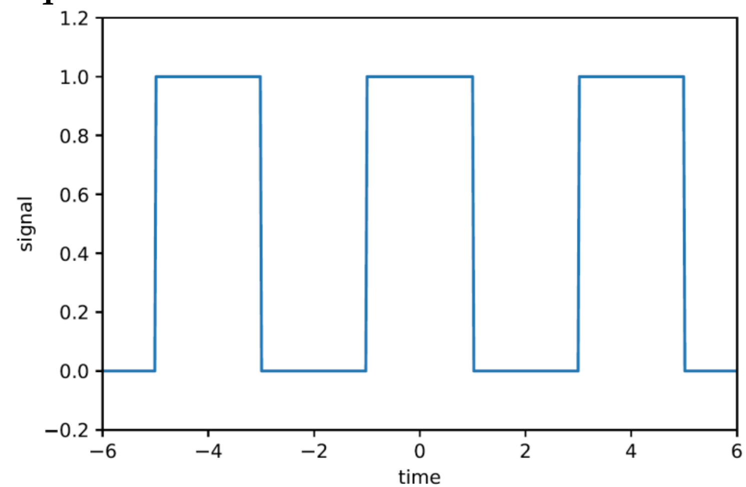
$$g(t) = \sum_n \left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T} \right)$$

$$\rightarrow g(t) = \int a(\omega)\cos\omega t + b(\omega)\sin\omega t \, d\omega,$$

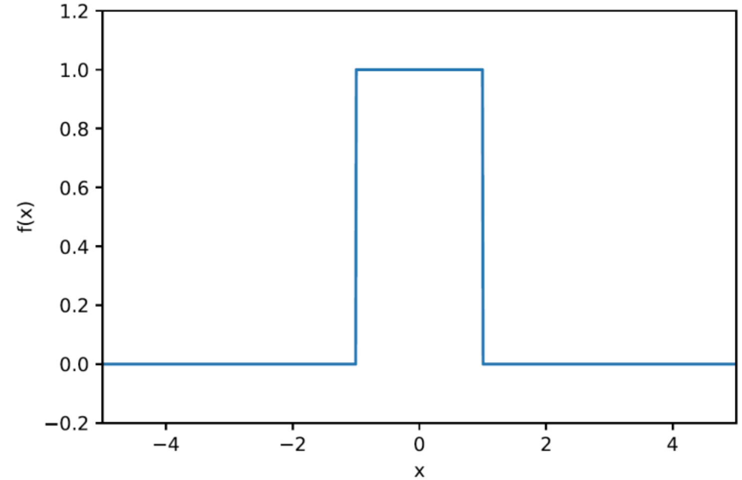
$$\text{or } g(t) = \frac{1}{2}\sum \mathrm{Re}\left[w_n \exp\left(\frac{2in\pi t}{T}\right)\right]$$

$$\rightarrow g(t) = \frac{1}{2\pi}\int w(\omega)\exp(i\omega t)\,d\omega.$$

- Square wave:



- Top hat:



Fourier series and transforms

- Conventional notation, is that Fourier transform of a function f(x) is written:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(x)\exp(-i\omega x)\,dx.$$

- And the original function can be represented using:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\exp(i\omega x)\,d\omega.$$

- Look at an example, the top hat.

$$\tilde{f}(\omega) = \int_{-1}^{1} (1)\exp(-i\omega x)\,dx$$

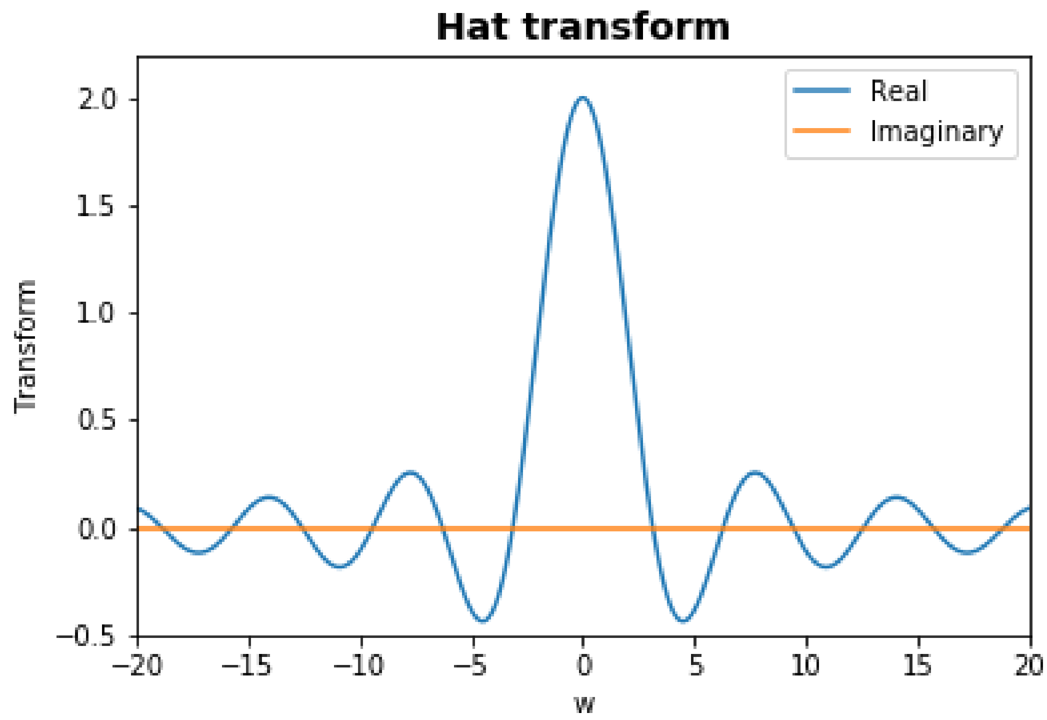
$$= -\frac{1}{i\omega}\exp(-i\omega x)\Big|_{-1}^{1}$$

$$= -\frac{1}{i\omega}\left(\exp(-i\omega) - \exp(i\omega)\right).$$

- This can be simplified:

$$-\frac{1}{i\omega}\left(\exp(-i\omega) - \exp(i\omega)\right)$$

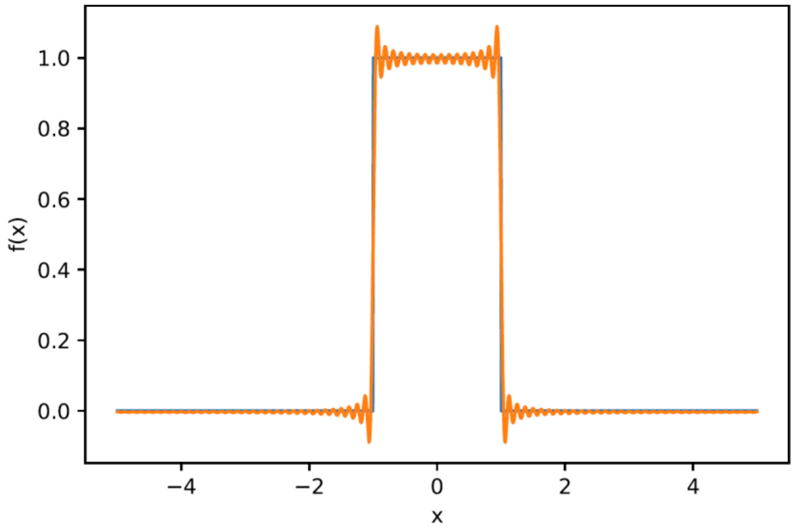
$$= \frac{2}{\omega}\left(\frac{\exp(i\omega) - \exp(-i\omega)}{2i}\right) = \frac{2}{\omega}\sin\omega.$$



Fourier series and transforms

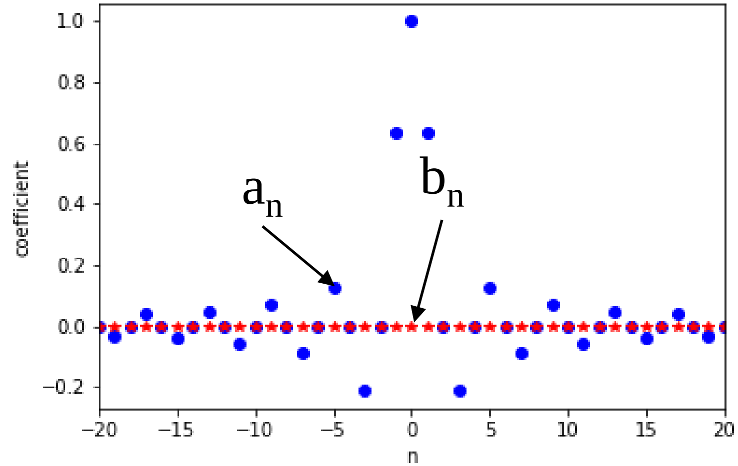
- Now use the transform to represent the original function:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp(i\omega x) d\omega.$$

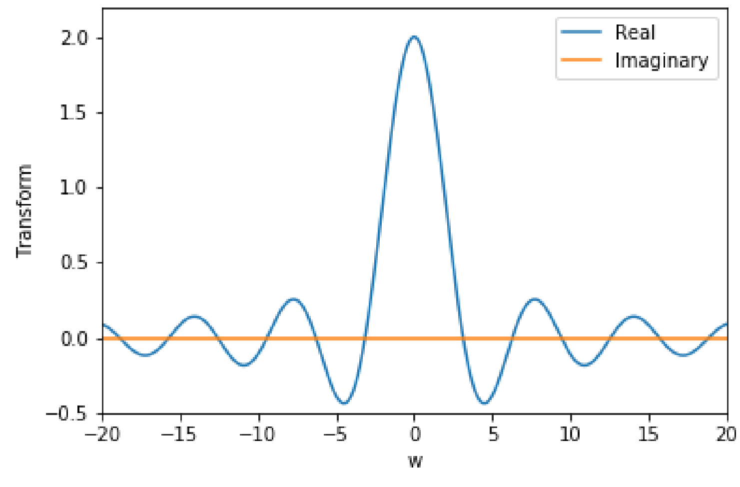


- Representation not perfect, because integration range reduced, equivalent to taking only first terms in the Fourier series.

- Fourier coefficients for square wave:



- Fourier transform for top hat:

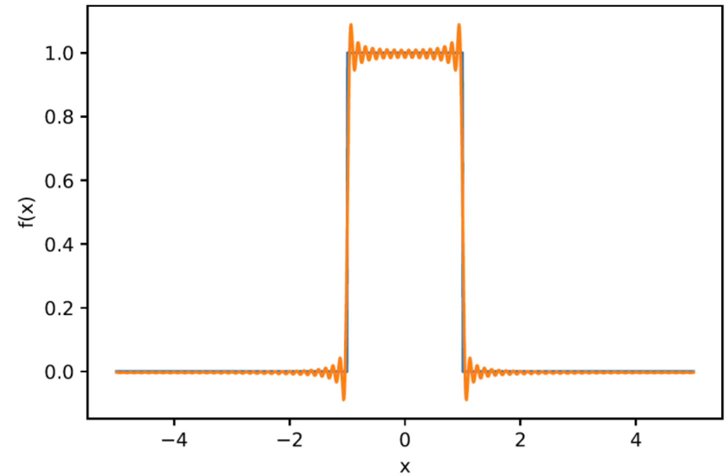


Using Fourier transforms

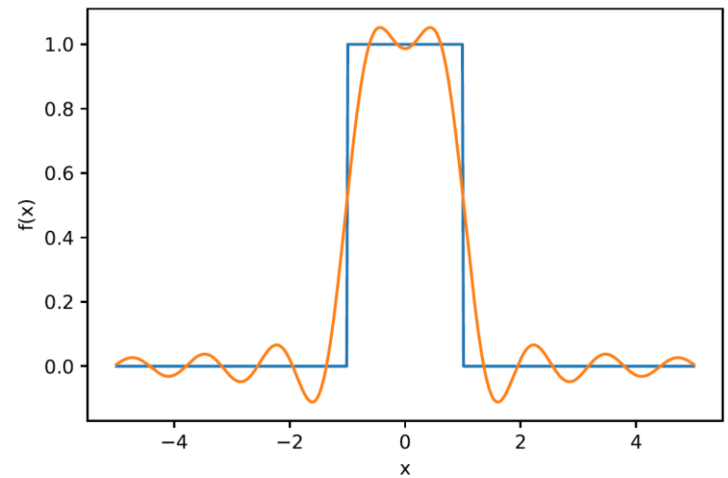
- Similar example as for Fourier series.
- How does an electronic pulse respond to passage through a high- or low-pass filter?
- Describe the pulse using a Fourier transform.
- Apply the frequency dependent function (filter) and evaluate the inverse Fourier transform.
- E.g. for low-pass filters, reduce range of integration

$$f(x) = \frac{1}{2\pi} \int_{-2\pi f_{top}}^{2\pi f_{top}} \tilde{f}(\omega) \exp(i\omega x) d\omega$$

- Input:



- Output:



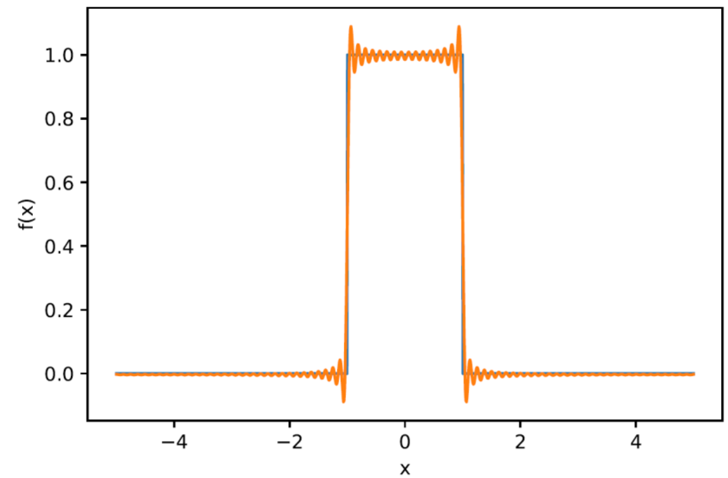
Using Fourier transforms

- For high-pass filter, omit central region of integration:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{-2\pi f_{bot}} \tilde{f}(\omega) \exp(i\omega x)\, d\omega + \frac{1}{2\pi} \int_{2\pi f_{bot}}^{\infty} \tilde{f}(\omega) \exp(i\omega x)\, d\omega$$

- For other filters, multiply transform by function that represents required effect.

- Input:



- Output: