Fourier series

- In this lecture we will:
  - See how to represent general periodic functions using Fourier series.
  - Look some more at odd and even functions.
  - Do some more examples.
- A comprehension question for this lecture:
  - Work out the Fourier series that describes the function:

    $$f(t) = -t \text{ for } -1 \le t < 1,$$

    $$f(t+2) = f(t) \text{ for all t.}$$

Functions with general period

- To represent a function with period T, we must scale the result derived for functions with period 2π.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}.$$

- The coefficients are found using:

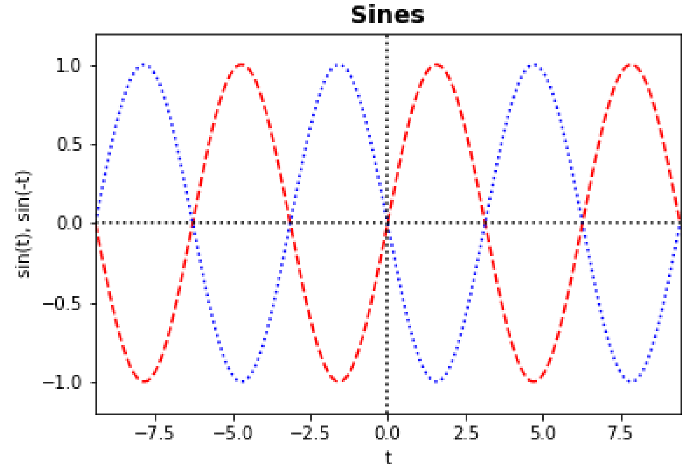
$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt,$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt,$$

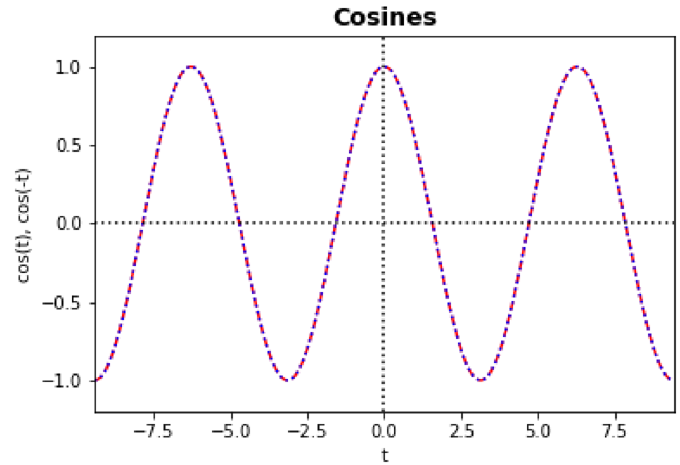
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt.$$

Odd and even functions

- Odd function, $f(x) = -f(-x)$, e.g. sine:



- Even function, $f(x) = f(-x)$, e.g. cosine:



Odd and even functions

- The integral of an odd function over a symmetric range [– T/2, T/2] is zero.
- "Obvious" from graph, but prove it!

$$\int_{-T/2}^{T/2} g(t)\,dt = \int_{-T/2}^{0} g(t)\,dt + \int_{0}^{T/2} g(t)\,dt.$$

- Put $t = -u \Rightarrow dt = -du$ in first integral.

$$\begin{aligned}\int_{-T/2}^{0} g(t)\,dt &= \int_{T/2}^{0} g(-u)(-du)\\ &= \int_{T/2}^{0} -g(u)(-du)\\ &= \int_{T/2}^{0} g(u)\,du\\ &= -\int_{0}^{T/2} g(u)\,du\\ &= -\int_{0}^{T/2} g(t)\,dt.\end{aligned}$$

- Hence:

$$\begin{aligned}\int_{-T/2}^{T/2} g(t)\,dt &= -\int_{0}^{T/2} g(t)\,dt + \int_{0}^{T/2} g(t)\,dt\\ &= 0.\end{aligned}$$

- If an even function, f(t), is multiplied by an odd function, sine, the result is an odd function.
- Hence:

$$\begin{aligned}b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt\\ &= 0.\end{aligned}$$

- Similarly, for an odd function g(t):

$$\begin{aligned}a_n &= \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos\frac{2n\pi t}{T}\,dt\\ &= 0.\end{aligned}$$

Odd and even functions

- The integrals used to determine the Fourier coefficients can be simplified if the integrand is even, e.g. for an even function f(t):

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt$$
$$= \frac{4}{T}\int_{0}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt.$$

- Even functions can be written:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T}.$$

- And odd functions:

$$g(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}.$$

Sawtooth function

- The sawtooth function is:

$$f(t) = t \text{ for } -1 \le t < 1,$$
$$f(t+2) = f(t) \text{ for all t.}$$

- Here, $T = 2$.
- Calculate the Fourier coefficients.
- f(t) is odd, so a_0 and all a_n are zero.
- Must calculate b_n, but can use fact that integrand is even (product of two odd functions):

$$b_n = \int_{-1}^{1} t\sin n\pi t\,dt$$
$$= 2\int_{0}^{1} t\sin n\pi t\,dt.$$

Sawtooth function

- $$b_n = 2\int_0^1 t\sin n\pi t\,dt$$
$$= 2\int_0^1 t\,d\left(\frac{-\cos n\pi t}{n\pi}\right)$$
$$= -\frac{2t\cos n\pi t}{n\pi}\bigg|_0^1 + 2\int_0^1 \frac{\cos n\pi t}{n\pi}\,dt$$
$$= -\frac{2\cos n\pi}{n\pi} + \frac{2}{n^2\pi^2}\sin n\pi t\bigg|_0^1$$
$$= -\frac{2\cos n\pi}{n\pi}$$
$$= -2\frac{(-1)^n}{n\pi}$$
$$= 2\frac{(-1)^{n+1}}{n\pi}.$$

- So:
$$f(t) = \frac{2}{\pi}\left(\sin \pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t - \ldots\right)$$