Series solution of differential equations
Legendre polynomials

- In this lecture we will:
  - Use the power series method to solve general differential equations.
  - Use the power series technique to solve Legendre's equation using Legendre polynomials.
  - Look at some properties and applications of Legendre polynomials.

- A comprehension question for this lecture:
  - Write $x^3 + 2x$ in terms of Legendre polynomials by using their orthonormality.

Power series solution of differential equations

- So far, we have found solutions for differential equations which have a number of specific forms.
- For general 1D differential equations, we can find a solution as a power series which will give us an approximation to the exact general solution for x close to a given value (often for x close to zero).
- For some equations, exact solutions can be found using the power series technique.
- Legendre's equation is one such case.

- Power series solution example.
- Find an approximate solution to the equation:

$$\frac{d^2y}{dx^2} + y = x\frac{dy}{dx}.$$

- Write down y as a polynomial:

$$y = a_0x^0 + a_1x^1 + a_2x^2 + \ldots + a_rx^r + \ldots$$

- Calculate needed derivatives:

$$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots + (r+1)a_{r+1}x^r + \ldots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots + (r+2)(r+1)a_{r+2}x^r + \ldots$$

Power series solution of differential equations

- Substitute polynomial and derivatives in differential equation.
- $y'' + y = xy'$ so:

$$2a_2 + 6a_3x + 12a_4x^2 + \ldots + (r+2)(r+1)a_{r+2}x^r + \ldots$$
$$+ a_0 + a_1x^1 + a_2x^2 + \ldots + a_rx^r + \ldots$$
$$= x(a_1 + 2a_2x + 3a_3x^2 + \ldots + (r+1)a_{r+1}x^r + \ldots)$$

- This must hold for all values of x, so coefficients of x^n on LH and RH sides must be the same for all n.
- Comparing powers of x^0:

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2}.$$

- Comparing powers of x^1:

$$6a_3 + a_1 = a_1 \Rightarrow a_3 = 0.$$

- And powers of x^r:

$$(r+2)(r+1)a_{r+2} + a_r = ra_r$$

$$\text{therefore } a_{r+2} = \frac{r-1}{(r+2)(r+1)}a_r.$$

Power series solution of differential equations

- Hence we can write down the polynomial incorporating the relationships between its coefficients:

$$y = a_0 + a_1 x - \frac{a_0}{2} x^2 + 0x^3 + \ldots$$

- Further coefficients can be found using the recurrence relationship.
- What are the values of the coefficients multiplying x^4 and x^5?
- There are two arbitrary constants, a_0 and a_1 (this is a second order equation!).
- These can be found using the initial conditions.
- For example, if $y(0) = 0$, we see $a_0 = 0$.
- Using this, and differentiating the polynomial solution, we see $y' = a_1 + \ldots$
- So if $y'(0) = 1$, this implies $a_1 = 1$.

Legendre's equation

- Legendre's equation is:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

- Crops up a lot in physics, in particular in quantum mechanics.
- Solve using a power series.

$$y=\sum_{r=0}^{\infty}a_r x^r \Rightarrow y'=\sum_{r=1}^{\infty}ra_r x^{r-1}$$

$$\text{and } y''=\sum_{r=2}^{\infty}r(r-1)a_r x^{r-2}.$$

- Hence

$$(1-x^2)\sum_{r=2}^{\infty}r(r-1)a_r x^{r-2}-2x\sum_{r=1}^{\infty}ra_r x^{r-1}$$

$$+n(n+1)\sum_{r=0}^{\infty}a_r x^r=0.$$

- Tidying up:

$$\sum_{r=2}^{\infty}r(r-1)a_r x^{r-2}-\sum_{r=2}^{\infty}r(r-1)a_r x^r-2\sum_{r=1}^{\infty}ra_r x^r$$

$$+n(n+1)\sum_{r=0}^{\infty}a_r x^r=0.$$

- Term in x^0:

$$2a_2x^0+n(n+1)a_0x^0=0.$$

- Term in x^1:

$$6a_3x^1-2a_1x^1+n(n+1)a_1x^1=0.$$

- Term in x^r:

$$(r+2)(r+1)a_{r+2}x^r-r(r-1)a_r x^r-2ra_r x^r$$

$$+n(n+1)a_r x^r=0$$

$$\Rightarrow (r+2)(r+1)a_{r+2}x^r=$$

$$\big(r(r-1)+2r-n(n+1)\big)a_r x^r$$

Legendre's equation

- From x^0 term:

$$a_2 = -\frac{n(n+1)}{2}a_0.$$

- From x^1 term:

$$a_3 = \frac{2-n(n+1)}{6}a_1.$$

- From x^r term:

$$(r+2)(r+1)a_{r+2} = \big(r(r+1)-n(n+1)\big)a_r$$

$$\Rightarrow a_{r+2} = \frac{r(r+1)-n(n+1)}{(r+2)(r+1)}a_r.$$

- Rewriting this:

$$a_{r+2} = \frac{r^2+r-n^2-n}{(r+2)(r+1)}a_r$$

$$= -\frac{(n+r+1)(n-r)}{(r+2)(r+1)}a_r.$$

- If we put $r = n$, we see $a_{n+2} = 0.$
- Hence $a_{n+4} = a_{n+6} = \ldots = 0.$
- So if n is even, the series starts at a_0 and stops at a_n.
- If n is odd, the series starts at a_1 and stops at a_n.
- In both cases, the solution is a finite *Legendre polynomial*.

Legendre polynomials

- The first few Legendre polynomials are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

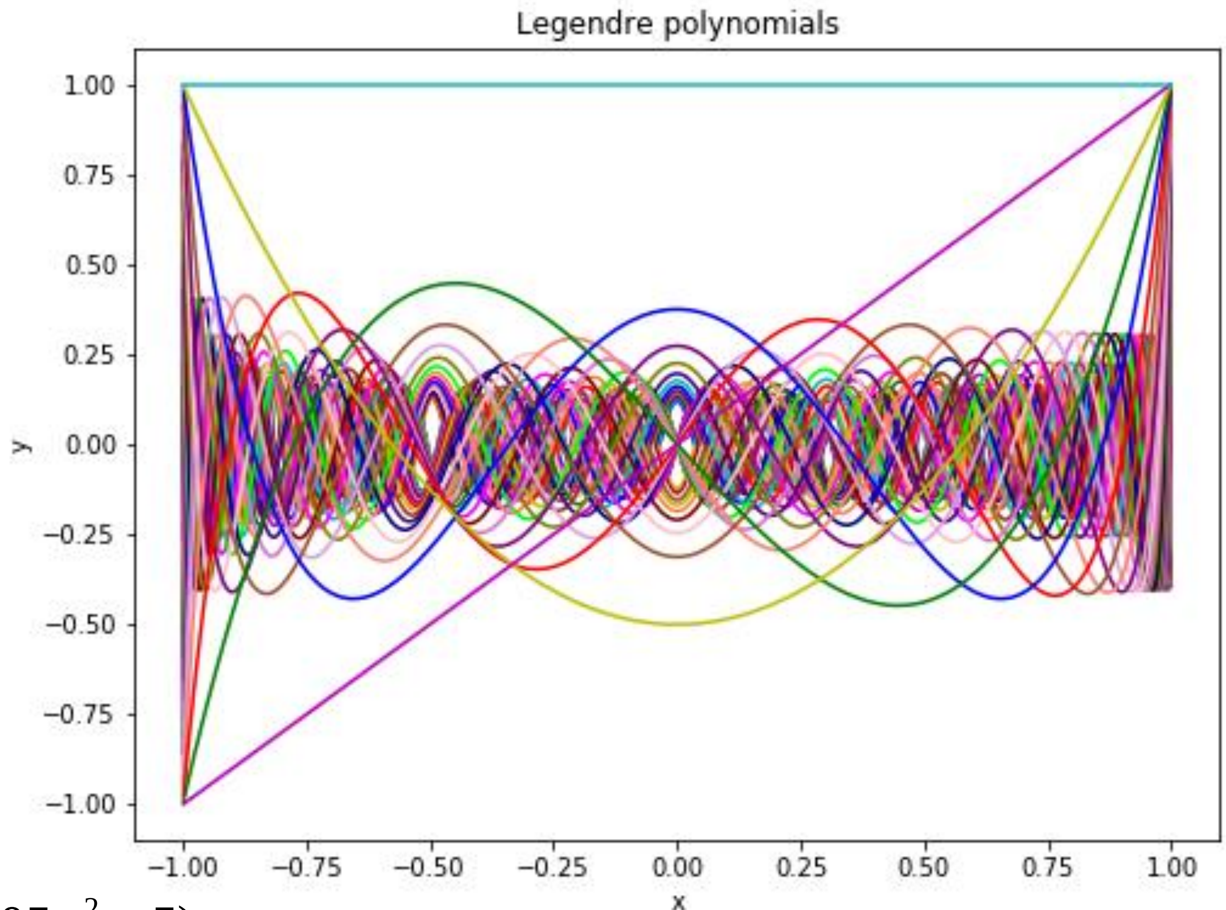
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

- Plot of first 45 Legendre polynomials:



Properties of Legendre polynomials

- Legendre polynomials have interesting properties, the most important being orthonormality:

$$\langle P_m(x), P_n(x) \rangle \equiv \int_{-1}^{1} P_m(x) P_n(x)\, dx$$
$$= 0,\ m \neq n$$

$$\langle P_n(x), P_n(x) \rangle \equiv \int_{-1}^{1} [P_n(x)]^2\, dx$$
$$= \frac{2}{2n+1}$$

- The operation $\langle\ \rangle$ is analogous to the scalar product (dot product) of two vectors.

- We can think of functions defined on the interval [-1, 1] as spanning an infinite vector space.
- One *basis* is formed by the *monomials* 1, x, x^2, x^3…
- The Legendre polynomials form another.
- For example, we can write:
  $$x^2 = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x).$$
- The values of c_0, c_1 and c_2 could be found by comparing coefficients of x on each side of the above equation.
- Alternatively, the function x^2 can be projected onto the Legendre polynomial “basis vectors” using $\langle\ \rangle$.

Properties of Legendre polynomials

- For example:

$$\langle x^2, P_2\rangle = \langle c_0P_0 + c_1P_1 + c_2P_2, P_2\rangle$$
$$= c_0\langle P_0, P_2\rangle + c_1\langle P_1, P_2\rangle + c_2\langle P_2, P_2\rangle$$
$$= c_2\langle P_2, P_2\rangle$$
$$= c_2\frac{2}{2\times 2+1} = \frac{2}{5}c_2.$$

- Cf. $\langle x^2, P_2\rangle = \int_{-1}^{1} x^2\frac{1}{2}(3x^2-1)\,dx$

$$= \frac{1}{2}\left[3\frac{x^5}{5} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{15}.$$

- Hence $\frac{2}{5}c_2 = \frac{4}{15}$ or $c_2 = \frac{2}{3}$.

- The Legendre polynomials can also be constructed by using their orthonormality properties…
- …and noting that:
  - $P_n(x)$ is of degree n.
  - The even $P_n(x)$ only contain even powers of x.
  - The odd $P_n(x)$ only contain odd powers of x.
- Suppose we know $P_1(x) = x$ and we want to find $P_3(x)$.
- Write $P_3(x) = ax^3 + bx$.

Properties of Legendre polynomials

- Using the results above we can write:

$$\langle P_3, P_1 \rangle = \int_{-1}^{1} (ax^3 + bx)x\,dx$$

$$= \left[a\frac{x^5}{5} + b\frac{x^3}{3} \right]_{-1}^{1} = 2\left(\frac{a}{5} + \frac{b}{3} \right).$$

- Now $\langle P_3, P_1 \rangle = 0$ so:

$$2\left(\frac{a}{5} + \frac{b}{3} \right) = 0 \Rightarrow b = -\frac{3}{5}a.$$

- Look at:

$$\langle P_3, P_3 \rangle = \frac{2}{2\times 3+1} = \frac{2}{7}.$$

- But: $\langle P_3, P_3 \rangle = \int_{-1}^{1} (ax^3 + bx)^2\,dx$

$$= \left[\frac{a^2}{7}x^7 + 2\frac{ab}{5}x^5 + \frac{b^2}{3}x^3 \right]_{-1}^{1}$$

$$= 2\left(\frac{a^2}{7} + \frac{2ab}{5} + \frac{b^2}{3} \right) = \frac{8}{25\times 7}a^2.$$

- Hence:

$$\frac{8}{25\times 7}a^2 = \frac{2}{7},\ a^2 = \frac{25}{4},\ a = \frac{5}{2} \text{ and } b = -\frac{3}{2}.$$

- By convention, the highest power has a positive coefficient.
- Putting this together, we have:

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$