Differential equations

- In this lecture we will:
  - Find out how to solve various types of inhomogeneous second order differential equation.
- Some comprehension questions for this lecture.
  - Find the general solution of the equations:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = x^2 + 5$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 5e^{-3x}$$

Inhomogeneous second order differential equations

- Here, we look at inhomogeneous (or non-homogeneous) second order differential equations, i.e. equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = f(x)$$

- The homogenous differential equation obtained by setting f(x) = 0,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0,$$

  with the same coefficients as the above, is called the complementary equation.

- Suppose the general solution (containing arbitrary constants) of the complementary equation is $y_c(x)$ and that a particular solution (no arbitrary constants) of the inhomogeneous equation is $y_p(x)$.
- We can then show that

$$y(x) = y_c(x) + y_p(x)$$

  is a general solution of the inhomogeneous equation.
- Do this by writing the solution of the complementary equation in the form $y_c(x) = y(x) - y_p(x)$.

Inhomogeneous second order differential equations

- Then, substituting for $y_c(x)$:

$$a\frac{d^2}{dx^2}(y-y_p)+b\frac{d}{dx}(y-y_p)+c(y-y_p)$$

$$=a\frac{d^2}{dx^2}y_c+b\frac{d}{dx}y_c+cy_c$$

$$\Rightarrow a\frac{d^2}{dx^2}y+b\frac{d}{dx}y+cy$$

$$-\left(a\frac{d^2}{dx^2}y_p+b\frac{d}{dx}y_p+cy_p\right)=0$$

$$\Rightarrow a\frac{d^2}{dx^2}y+b\frac{d}{dx}y+cy$$

$$=a\frac{d^2}{dx^2}y_p+b\frac{d}{dx}y_p+cy_p=f(x)$$

- Hence, if we can find a general solution of the complementary equation, $y_c(x)$, and a particular solution (particular integral) of the inhomogeneous equation, their sum will be a general solution of the inhomogeneous equation.
- We already know how to find solutions of homogeneous equations with constant coefficients.
- How can we find particular solutions of inhomogeneous equations (again restricted to constant coefficients)?
- Educated guesswork...also known as the method of undetermined coefficients.

Inhomogeneous second order differential equations

- An example:
- Find the general solution of the equation:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=x^2$$

- The complementary equation is...

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

- ...and the associated auxiliary equation:

$$m^2+m-2=0$$

$$\Rightarrow (m-1)(m+2)=0$$

$$m_1=1,\ \ m_2=-2$$

- Hence the general solution of the complementary equation is

$$y_c(x)=C_1e^x+C_2e^{-2x}$$

- Since f(x) = x^2 and differentiating this will give both a term in x and a constant, we try the particular solution

$$y_p(x)=Ax^2+Bx+C$$

- The values of A, B and C can be determined by substituting into the inhomogeneous equation.
- We need y_p and its differentials:

$$\frac{dy_p}{dx}=2Ax+B \text{ and } \frac{d^2y_p}{dx^2}=2A.$$

Inhomogeneous second order differential equations

- Hence:

$$(2A)+(2Ax+B)$$
$$-2\left(Ax^2+Bx+C\right)=x^2$$
$$\Rightarrow -2Ax^2+(2A-2B)x$$
$$+2A+B-2C=x^2$$

- For this to hold for all x, must have:

$-2A=1$ [coefficents of x^2]

$2A-2B=0$ [coefficents of x]

and $2A+B-2C=0$ [constants].

- Hence:

$A=-1/2,\ 2B=2A\Rightarrow B=-1/2$ and

$$C=\frac{2A+B}{2}=-\frac{3}{4}.$$

- Our particular solution is thus:

$$y_p=-\frac{1}{2}x^2-\frac{1}{2}x-\frac{3}{4}.$$

- And the general solution is:

$$y(x)=y_c(x)+y_p(x)$$
$$=C_1e^x+C_2e^{-2x}-\frac{1}{2}x^2-\frac{1}{2}x-\frac{3}{4}.$$

- Note, we still have two arbitrary constants, the values of which can be determined using initial conditions.
- This illustrates how inhomogeneous differential equations can be solved if $f(x)$ is a polynomial.
- There is one possible difficulty...

Inhomogeneous second order differential equations

- Find a particular solution to the differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 5$$

- Try $y_p = C$

$$\Rightarrow \frac{dy_p}{dx} = 0$$

- Cannot substitute to work out coefficients.
- Must modify trial function to ensure we get required behaviour.
- In general, multiply y_p by x, e.g. if trial function $x^2 + 2x + 1$ doesn't work, try $x(x^2 + 2x + 1)$.
- Example here, try $y_p(x) = Cx$.
- Then have:

$$\frac{dy_p}{dx} = C \text{ and } \frac{d^2y_p}{dx^2} = 0.$$

- Hence C = 5 and $y_p(x) = 5x$.
- The auxiliary equation is $m(m+1) = 0$ so the general solution of the complementary equation is:

$$y_c = C_1e^0 + C_2e^{-x} = C_1 + C_2e^{-x}$$

- The general solution of the inhomogeneous equation is therefore

$$y(x) = C_1 + C_2e^{-x} + 5x$$

- C_1 and C_2 can then be determined using the initial conditions.

Inhomogeneous second order differential equations

- Now consider case that $f(x)$ is of the form $Ae^{\gamma x}$.
- The trial function depends on the roots of the auxiliary equation.
  - If $m_1, m_2 \neq \gamma$, try $y_p = Ae^{\gamma x}$.
  - If $m_1 = \gamma$, $m_2 \neq \gamma$, try $y_p = Axe^{\gamma x}$.
  - If $m_1 = m_2 = \gamma$, try $y_p = Ax^2e^{\gamma x}$.
- Example:
- Find a particular solution to:
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 3e^{-x}$$
- The auxiliary equation
$$m^2 + 2m + 1 = 0$$
$$\Rightarrow (m+1)^2 = 0 \text{ and } m_1 = m_2 = -1$$
- Hence we try:
$$y_p = Ax^2e^{-x}$$
$$\frac{dy_p}{dx} = -Ax^2e^{-x} + 2Axe^{-x}$$
$$\frac{d^2y_p}{dx^2} = Ax^2e^{-x} - 2Axe^{-x} - 2Axe^{-x} + 2Ae^{-x}$$
$$= Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x}$$
- Substituting gives:
$$\left(Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x}\right) +$$
$$2\left(-Ax^2e^{-x} + 2Axe^{-x}\right) + Ax^2e^{-x} = 3e^{-x}$$
- Similar to previous case, compare coefficients, in this case of e^{-x}, xe^{-x} and x^2e^{-x}, to determine A.

Inhomogeneous second order differential equations

- Hence $2Ae^{-x} = 3e^{-x}$ and $A = 3/2$.
- The solution of the complementary equation is $y_c = C_1e^{-x} + C_2xe^{-x}$ giving a general solution of the inhomogeneous equation
$$y = C_1e^{-x} + C_2xe^{-x} + \frac{3}{2}x^2e^{-x}.$$