Differential equations

- In this lecture we will:
  - Look at second order homogeneous differential equations.
  - Introduce the auxiliary equation and determine its roots.
  - Find out how to solve the homogeneous second order differential equation in the case that the roots of the auxiliary equation are:
    - Real and different.
    - The same.
    - Complex conjugate.

- Some comprehension questions for this lecture.
  - Write down the general form of a homogeneous second order differential equation with constant coefficients.
  - Solve the initial value problem:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0,$$

with $y(1) = 1$ and $\frac{dy(1)}{dx} = 1$

Homogeneous second order differential equations

- Consider second order homogeneous differential equations of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

- The coefficients a, b and c are all constants.
- Try to find a solution of the form

$$y=e^{mx}.$$

- Differentiating this gives:

$$\frac{dy}{dx}=me^{mx} \text{ and } \frac{d^2y}{dx^2}=m^2e^{mx}.$$

- Substituting into the original equation we have:

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0,$$

$$\text{or } e^{mx}(am^2+bm+c)=0$$

- Now e^{mx} cannot be zero, so:

$$am^2+bm+c=0.$$

- This is called the auxiliary equation.
- The above implies that $y = e^{mx}$ is a solution of the differential equation iff (if and only if) m takes one of the values:

$$m_1=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$m_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Homogeneous second order differential equations

- When the discriminant $b^2 - 4ac > 0$, m_1 and m_2 are real and distinct.
- When $b^2 - 4ac = 0$, the roots are real and equal.
- When $b^2 - 4ac < 0$, the roots are complex conjugate numbers.

- The principle of superposition:
- Supposing we have two solutions of our homogeneous second order differential equation, $y_1(x)$ and $y_2(x)$.
- The sum $C_1y_1(x) + C_2y_2(x)$ is also a solution of the equation.

- Prove this:

$$a\frac{d^2}{dx^2}\left(C_1y_1 + C_2y_2\right) +$$

$$b\frac{d}{dx}\left(C_1y_1 + C_2y_2\right) + c\left(C_1y_1 + C_2y_2\right)$$

$$= aC_1\frac{d^2y_1}{dx^2} + aC_2\frac{d^2y_2}{dx^2} +$$

$$bC_1\frac{dy_1}{dx} + bC_2\frac{dy_2}{dx} + cC_1y_1 + cC_2y_2$$

$$= C_1\left(a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right) +$$

$$C_2\left(a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2\right)$$

$$= 0.$$

Homogeneous second order differential equations

- Consider various possibilities for the solutions of the auxiliary equation.
- If we have distinct roots, $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent solutions of our differential equation.
- The functions $y_1(x)$ and $y_2(x)$ are linearly independent if one is not just a multiple of the other, that is: $y_2(x) \neq k y_1(x)$.
- Hence, by the superposition principle, a general solution is: $y(x) = Ae^{m_1 x} + Be^{m_2 x}$.

- Example:
- Find a general solution of: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0.$
- The auxiliary equation is:

$$m^2 + 5m - 6 = 0$$
$$\Rightarrow (m+6)(m-1) = 0$$
$$\Rightarrow m_1 = 1 \text{ and } m_2 = -6.$$

- Hence a general solution to the equation is: $y(x) = Ae^{x} + Be^{-6x}$.

Homogeneous second order differential equations

- Another example, solve the initial value problem

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-y=0$$

$$y(0)=0$$

$$\frac{dy(0)}{dx}=-1 \text{ or } y'(0)=-1.$$

- The auxiliary equation is:

$$m^2+2m-1=0$$

$$\Rightarrow m_1=\frac{-2+\sqrt{8}}{2}=-1+\sqrt{2}$$

$$\text{and } m_2=\frac{-2-\sqrt{8}}{2}=-1-\sqrt{2}.$$

- A general solution is

$$y(x)=Ae^{\left(-1+\sqrt{2}\right)x}+Be^{\left(-1-\sqrt{2}\right)x}.$$

- The initial conditions can be used to determine A and B:

$$y(0)=Ae^0+Be^0$$

$$\Rightarrow 0=A+B \text{ or } A=-B$$

$$\frac{dy}{dx}=A\left(-1+\sqrt{2}\right)e^{\left(-1+\sqrt{2}\right)x}+B\left(-1-\sqrt{2}\right)e^{\left(-1-\sqrt{2}\right)x}$$

$$\frac{dy(0)}{dx}=\left(-1+\sqrt{2}\right)A+\left(-1-\sqrt{2}\right)B$$

$$-1=\left(-1+\sqrt{2}\right)A-\left(-1-\sqrt{2}\right)A$$

$$\Rightarrow -1=2\sqrt{2}A \text{ or } A=-\frac{1}{2\sqrt{2}}.$$

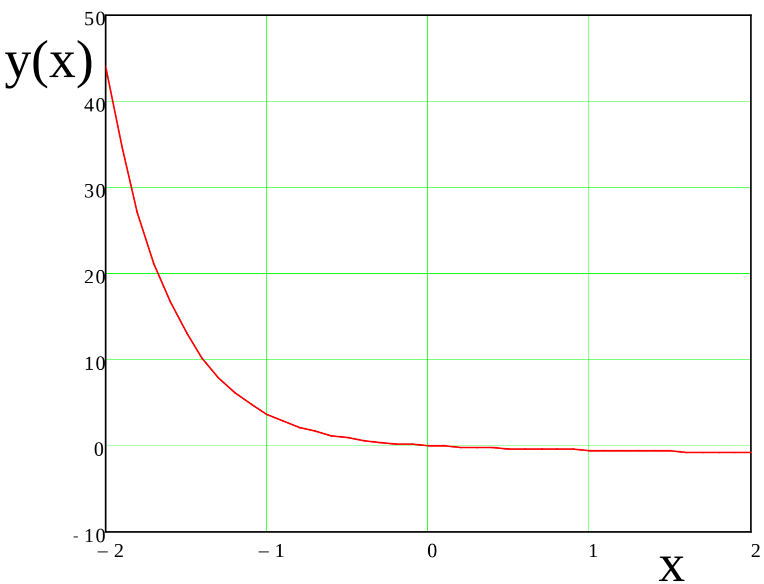
Homogeneous second order differential equations

- Rewriting:

$$A = -\frac{\sqrt{2}}{4} \text{ and } B = \frac{\sqrt{2}}{4}.$$

- Putting this together:

$$y(x) = -\frac{\sqrt{2}}{4}e^{(-1+\sqrt{2})x} + \frac{\sqrt{2}}{4}e^{(-1-\sqrt{2})x}.$$



- If the roots of auxiliary equation are the same (m), we can use $y = e^{mx}$ and $y = xe^{mx}$ as two linearly independent solutions of the differential equation.
- Example:
- Find a general solution of:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

- Auxiliary equation:

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow (m+2)^2 = 0$$

$$m_1 = m_2 = -2.$$

Homogeneous second order differential equations

- General solution therefore:

$$y(x) = Ae^{-2x} + Bxe^{-2x}$$

- What if the auxiliary equation has complex conjugate roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$?
- Then:

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right) \\ &= e^{\alpha x}\begin{pmatrix} A\left(\cos(\beta x) + i\sin(\beta x)\right) \\ + B\left(\cos(-\beta x) + i\sin(-\beta x)\right) \end{pmatrix} \\ &= e^{\alpha x}\begin{pmatrix} (A+B)\cos(\beta x) \\ + i(A-B)\sin(\beta x) \end{pmatrix}. \end{aligned}$$

- Writing P = A + B and Q = i(A – B), we then have the general solution:

$$y(x) = e^{\alpha x}\left(P\cos(\beta x) + Q\sin(\beta x)\right).$$

- Example:
- Find the general solution of:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0.$$

- Auxiliary equation:

$$m^2 + 2m + 4 = 0$$

$$m_1 = \frac{-2+\sqrt{4-16}}{2},\ m_2 = \frac{-2-\sqrt{4-16}}{2}$$

$$\Rightarrow m_1 = -1+\sqrt{-3},\ m_2 = -1-\sqrt{-3}$$

$$\text{or } m_1 = -1+i\sqrt{3} \text{ and } m_2 = -1-i\sqrt{3}.$$

Homogeneous second order differential equations

- So, with $\alpha = -1$ and $\beta = \sqrt{3}$:
  $y = Pe^{-x}\cos\sqrt{3}x + Qe^{-x}\sin\sqrt{3}x.$
- Another example:
- Solve the initial value problem:
  $$\frac{d^2y}{dx^2} + 4y = 0,\ y(0) = 1,\ \frac{dy(0)}{dx} = 0.$$
- Auxiliary equation:
  $m^2 + 4 = 0$
  $\Rightarrow m_1 = 2i,\ m_2 = -2i.$
- Hence $\alpha = 0,\ \beta = 2.$
- This gives:
  $y = P\cos 2x + Q\sin 2x,$
  $$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x.$$
- Using the initial conditions we have:
  $y(0) = P = 1,$
  $$\frac{dy(0)}{dx} = 2Q = 0, \text{ so } Q = 0.$$
- The required solution is therefore:
  $y(x) = \cos 2x.$