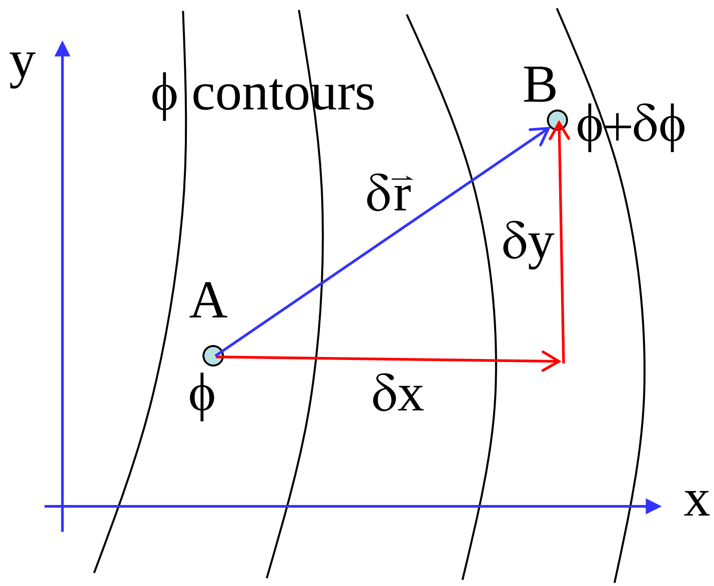
Vector calculus

- In this lecture we will:
  - Sketch out how we can derive a potential from a field using line integrals.
  - Do an example to check it works!
  - Look at a physical example: deriving the electric potential from the electric field.
  - Mention a caveat: there are some fields that cannot be derived from potentials.
- Some comprehension questions for this lecture.
  - What is the potential associated with the field:

$$\vec{E}(x,y,z) = \begin{pmatrix} yz + 2xy \\ x^2 + xz + z^2 \\ xy + 2yz \end{pmatrix}?$$

Deriving a potential from a field

- We have seen that we can get a field from a potential: $\vec{F}(x,y,z) = \nabla\phi(x,y,z)$.
- Suppose we have a field $\vec{F}(x,y)$, can we derive from this the associated potential $\phi(x,y)$?
- Illustrate idea in 2D (more formal proof in text books!).



- Consider stepping from A to B in the scalar field $\phi(x,y)$.
- Change in ϕ is $\delta\phi$, given by slope in direction of movement and step length.
- For step δx in x direction:

$$\delta\phi_x \approx \frac{\partial\phi(x,y)}{\partial x}\delta x = F_x \delta x.$$

- For subsequent step δy in y direction

$$\delta\phi_y \approx \frac{\partial\phi(x+\delta x,y)}{\partial y}\delta y$$

$$\approx \frac{\partial\phi(x,y)}{\partial y}\delta y \approx F_y \delta y.$$

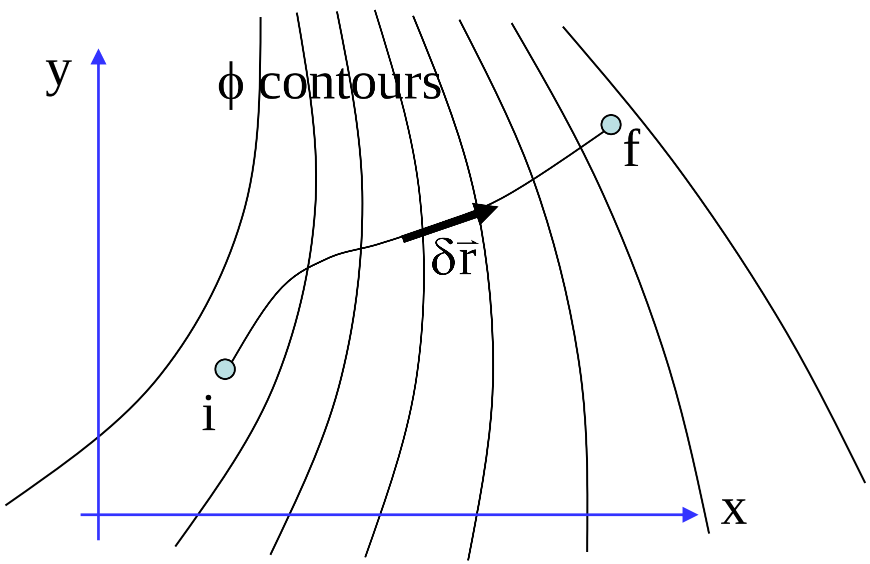
- If step in x then y, $\delta\phi \approx \delta\phi_x + \delta\phi_y$.

Deriving a potential from a field

- Rewriting this:

  $\delta\phi \approx F_x\,\delta x + F_y\,\delta y.$

- Now take n steps from initial position i to final position f:



- The total change in φ is then

$$\sum_n \delta\phi_n \approx \sum_n F_x(x_n, y_n)\,\delta x_n + F_y(x_n, y_n)\,\delta y_n$$

- Taking the limit of infinitely many infinitely small steps:

$$\int d\phi = \int_C F_x(x,y)\,dx + F_y(x,y)\,dy$$

$$\phi = \phi(x_i, y_i) + \int_C \left(F_x, F_y\right)\cdot\left(dx, dy\right)$$

$$= \phi(x_i, y_i) + \int_C \vec{F}\cdot d\vec{r}$$

- The subscript C tells us to move along curve from i to f.
- If start at (0, 0) and move to (x, y) we have “climbed” φ(x, y) - φ(0, 0).

Deriving a potential from a field

- Example:
- Field $\vec{E}(x,y) = \begin{pmatrix} 2x+y & x \end{pmatrix}$.
- Find the associated potential,

$$\phi = \phi_0 + \int_C \vec{E}\cdot d\vec{r}$$

- Integrate along $\vec{r}(t) = \begin{pmatrix} x(t) & y(t) \end{pmatrix}$

$$= \begin{pmatrix} xt & yt \end{pmatrix},\ t = 0 \ldots 1.$$

- Then:

$$\phi(x,y) = \phi_0 + \int_0^1 \vec{E}\big(x(t),y(t)\big)\cdot\frac{d\vec{r}(t)}{dt}dt$$

$$= \phi_0 + \int_0^1 E_x\big(x(t),y(t)\big)\frac{dx(t)}{dt}dt$$

$$+ \int_0^1 E_y\big(x(t),y(t)\big)\frac{dy(t)}{dt}dt$$

- Using $\dfrac{dx(t)}{dt} = \dfrac{d\,xt}{dt} = x$ and $\dfrac{dy(t)}{dt} = y$,

$$\phi = \phi_0 + \int_0^1 [2(xt)+(yt)]\times x\,dt + \int_0^1 (xt)\times y\,dt$$

$$= \phi_0 + \frac{t^2}{2}\times(2x^2+xy) + \frac{t^2}{2}\times xy\Big|_0^1$$

$$= \phi_0 + x^2 + \frac{xy}{2} + \frac{xy}{2}$$

$$= \phi_0 + x^2 + xy.$$

- Check:

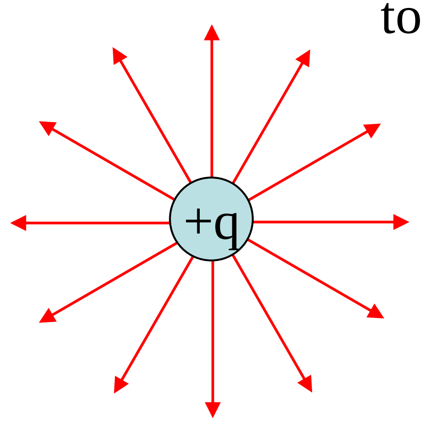
$$\nabla\phi = \begin{pmatrix} \dfrac{\partial}{\partial x}x^2 + xy + \phi_0 \\ \dfrac{\partial}{\partial y}x^2 + xy + \phi_0 \end{pmatrix} = \begin{pmatrix} 2x+y \\ x \end{pmatrix} = \vec{E}$$

Electric potential from electric field

- Electric field due to point charge given by:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\begin{pmatrix} \dfrac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\ \dfrac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\ \dfrac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \end{pmatrix}.$$

- Write $\dfrac{q}{4\pi\varepsilon_0} = K.$



- Using line integral method:

$$\phi = \phi_0 + \int_c \vec{E}\cdot d\vec{r}$$

$$= \phi_0 + \int_0^1 E_x \frac{dx}{dt}dt + \int_0^1 E_y \frac{dy}{dt}dt + \int_0^1 E_z \frac{dz}{dt}dt$$

with $\vec{r}(t) = \begin{pmatrix} xt \\ yt \\ zt \end{pmatrix}$ and taking the path

to be from $t = 0$ to $t = 1$ as before.

Electric potential from electric field

- Look at E_x integral:

$$\int_0^1 E_x \frac{dx}{dt}\,dt = K\int_0^1 \frac{xt}{\left((xt)^2+(yt)^2+(zt)^2\right)^{\frac{3}{2}}}\,x\,dt$$

$$= K\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\int_0^1 \frac{t}{\left(t^2\right)^{\frac{3}{2}}}\,dt$$

$$= K\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\int_0^1 \frac{1}{t^2}\,dt$$

$$= K\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left.\frac{-1}{t}\right|_0^1.$$

- There is a problem, can't evaluate one limit of integral: $\vec{E}$ infinite at origin!

- One solution is to change the path.
- Move from point at infinity to position (x, y, z) then have:

$$\int_\infty^1 E_x \frac{dx}{dt}\,dt = K\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left.\frac{-1}{t}\right|_\infty^1$$

$$= -K\frac{x^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}.$$

- Repeat for E_y and E_z and add results:

$$\phi = \phi_\infty - K\frac{\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{r}.$$

- Note minus sign, not present when "physics convention" used, we have decided $\vec{E} = -\nabla\phi$ not $\vec{E} = \nabla\phi$.

Caveat: fields that are not derivable from potentials

- Recall from lecture 6: $\nabla \times (\nabla \phi) = 0.$
- Hence, a vector field derived from a potential, e.g. $\vec{E} = \nabla\phi,$ must always satisfy $\nabla \times \vec{E} = 0.$
- Conversely, a field for which the curl is not zero cannot be derived from a potential.
- E.g. $\vec{F} = \begin{pmatrix} yz \\ xz - 3yz \\ xy + 2z \end{pmatrix}.$
- Using our prescription…

$$\phi = \phi_0 + \int_0^1 yt\,zt\,x\,dt + \int_0^1 (xt\,zt - 3yt\,zt)y\,dt$$

$$+\int_0^1 (xt\,yt + 2zt)z\,dt.$$

- So: $\phi = \phi_0 + xyz\int_0^1 t^2\,dt + (xyz - 3y^2z)\int_0^1 t^2 dt$

$$+xyz\int_0^1 t^2 dt + 2z^2\int_0^1 t\,dt$$

$$= \phi_0 + xyz\frac{t^3}{3} + (xyz - 3y^2z)\frac{t^3}{3}\Bigg|_0^1$$

$$+xyz\frac{t^3}{3} + 2z^2\frac{t^2}{2}\Bigg|_0^1.$$

$$\Rightarrow \phi = \phi_0 + xyz - y^2z + z^2.$$

- Now calculate field:

$$\nabla(xyz - y^2z + z^2) = \begin{pmatrix} yz \\ xz - 2yz \\ xy - y^2 + 2z \end{pmatrix} \neq \vec{F}.$$