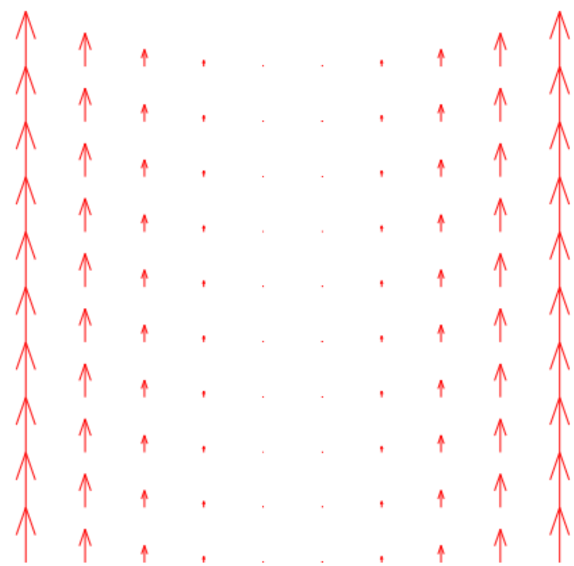
Vector calculus

- In this lecture we will:
  - Define the curl of a vector field.
  - Look at some examples to try and gain some insight into what the curl represents.
  - Discuss the curl of the electric and magnetic fields.
- Some comprehension questions for this lecture.
  - Indicate where the curl will be positive below.



  - Calculate the curl of the field:

$$\vec{F}(x,y,z) = \begin{pmatrix} y & xy & 0 \end{pmatrix}$$

Curl of a vector field

- The curl of a vector field is defined by the equation:

$$\nabla \times \vec{E}(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

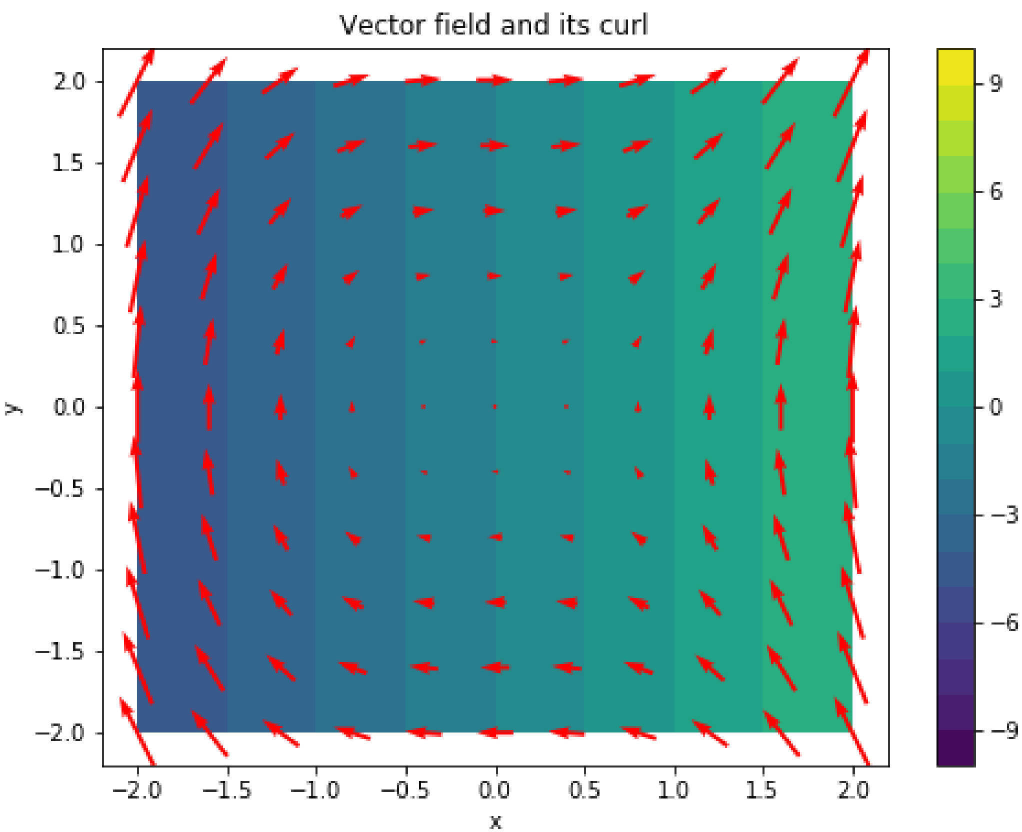
$$= \begin{pmatrix} \frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y \\ \frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z \\ \frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x \end{pmatrix}$$

- The curl of a vector field is a vector field.
- Can think of curl as cross product of ∇ operator and vector.
- Look at an example (with z component zero so we can plot it!).
- $\vec{F}(x,y,z) = \begin{pmatrix} y \\ x^2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y \\ \frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z \\ \frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x-1 \end{pmatrix}$$

Curl of vector field

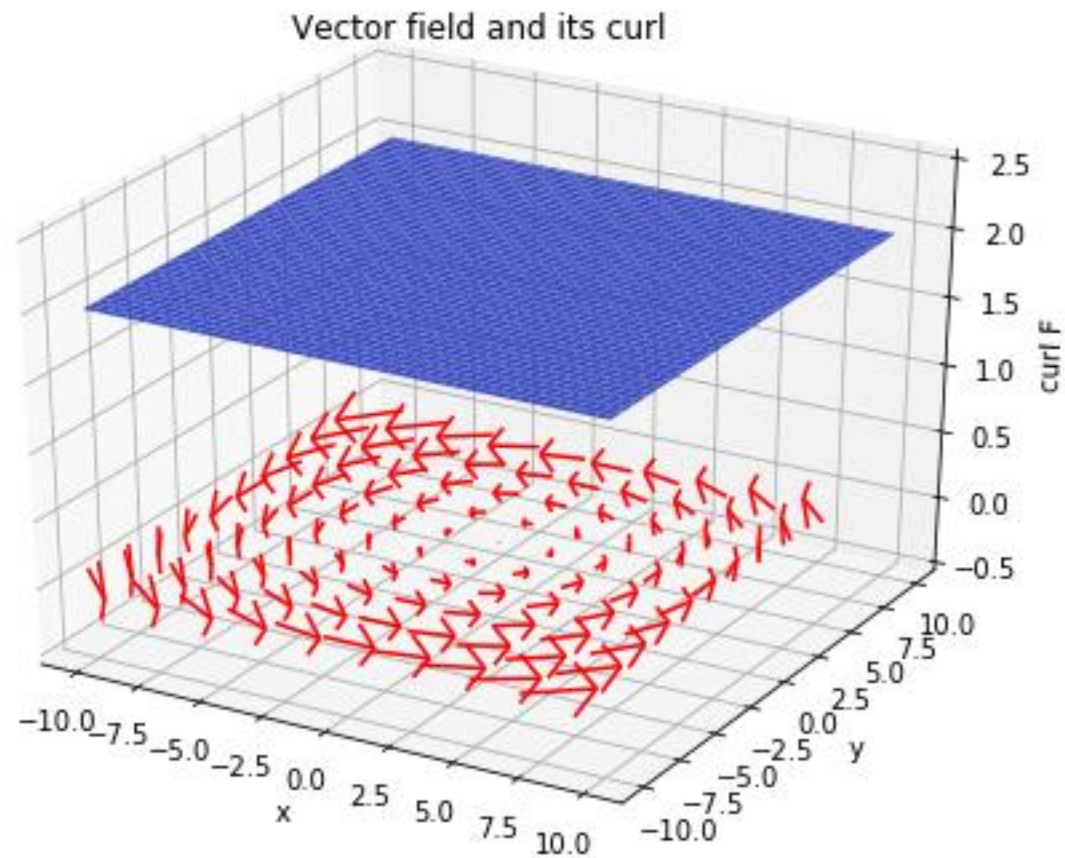
- Plot the x and y components of $\vec{F}$ as a vector field and the curl as a contour plot (shaded):



- What does the curl tell us about the field?
- Again, the name gives as a hint!
- (A further hint is that the curl of a field is sometimes called the rotation.)
- See that the curl is positive where a small object “dropped into the field” would rotate in an anticlockwise direction and negative where it would rotate in a clockwise direction.

Curl of vector field

- Now define field which has constant angular velocity and look at its curl.
- Use v = rω and set ω = 1, implies:
- $\vec{v}(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$
- Hence $\nabla \times \vec{v}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$
- Magnitude of curl is twice the angular velocity.
- Direction of curl is that of axis about which rotation occurs.
- Plot these quantities:



Calculate a curl

- Calculate the curl of the field:
  $\vec{F}(x, y, z) = \left(3\sin x \quad 2\cos x \quad -z^2\right).$

- Determine the value of
  $\nabla \times \vec{F}(-\tfrac{\pi}{2}, \tfrac{\pi}{2}, 3)$

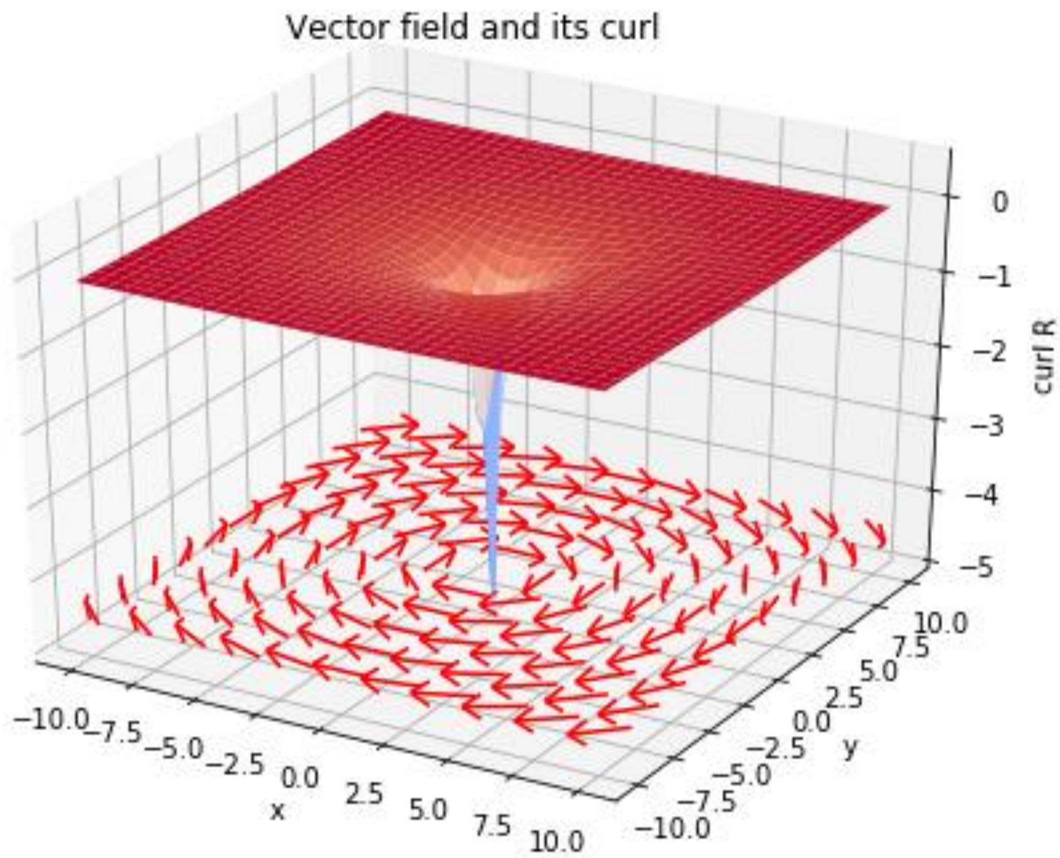
Curl of vector field

- Construct further examples:

- $$\vec{R}(x,y,z) = \begin{pmatrix} \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-x}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix}$$

- $$\nabla \times \vec{R}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{\sqrt{x^2+y^2}} \end{pmatrix}$$

- Plotting these quantities:

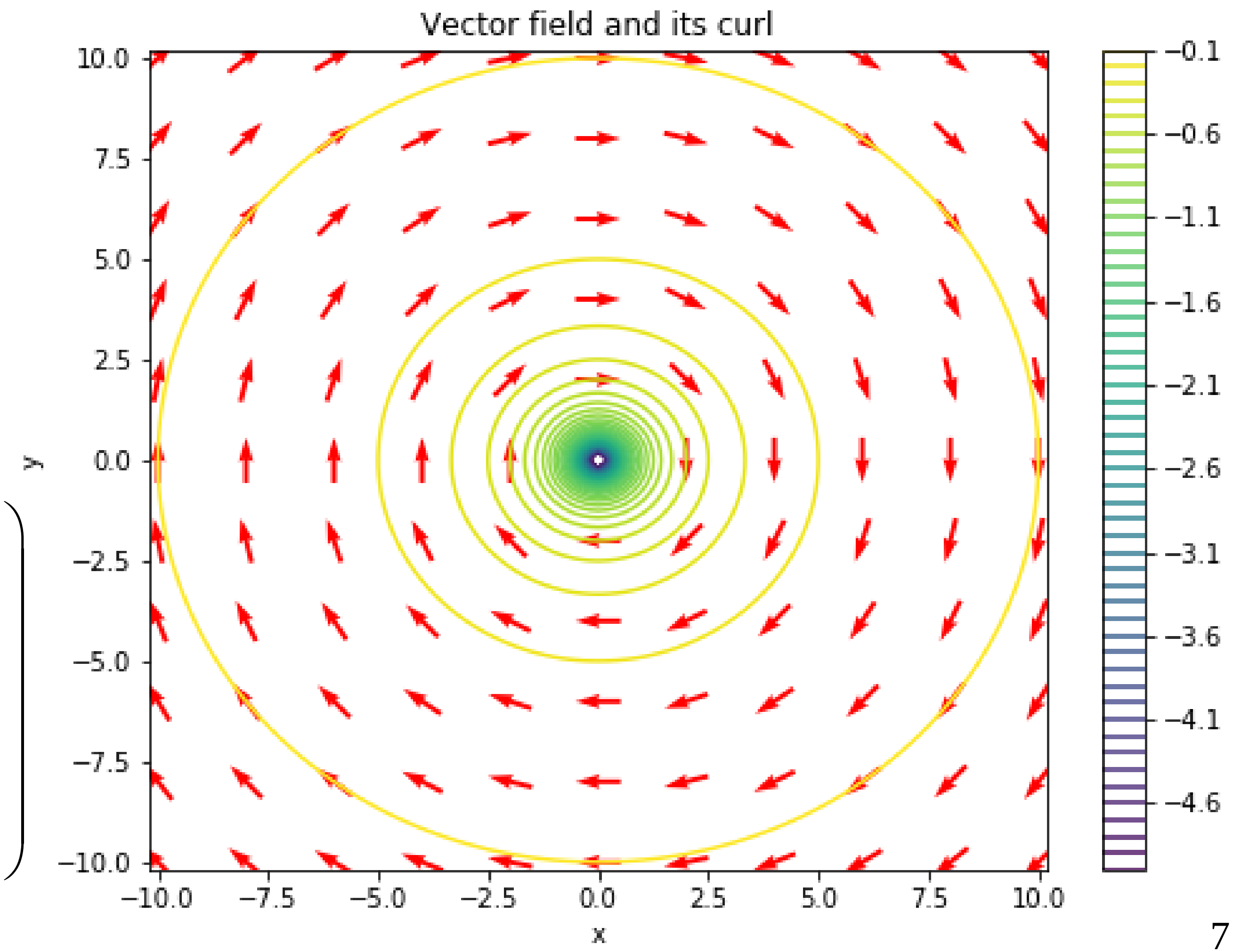


Curl of vector field

- Now using a contour plot for the curl:

- $$\vec{R}(x,y,z) = \begin{pmatrix} \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-x}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix}$$

- $$\nabla \times \vec{R}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{\sqrt{x^2+y^2}} \end{pmatrix}$$



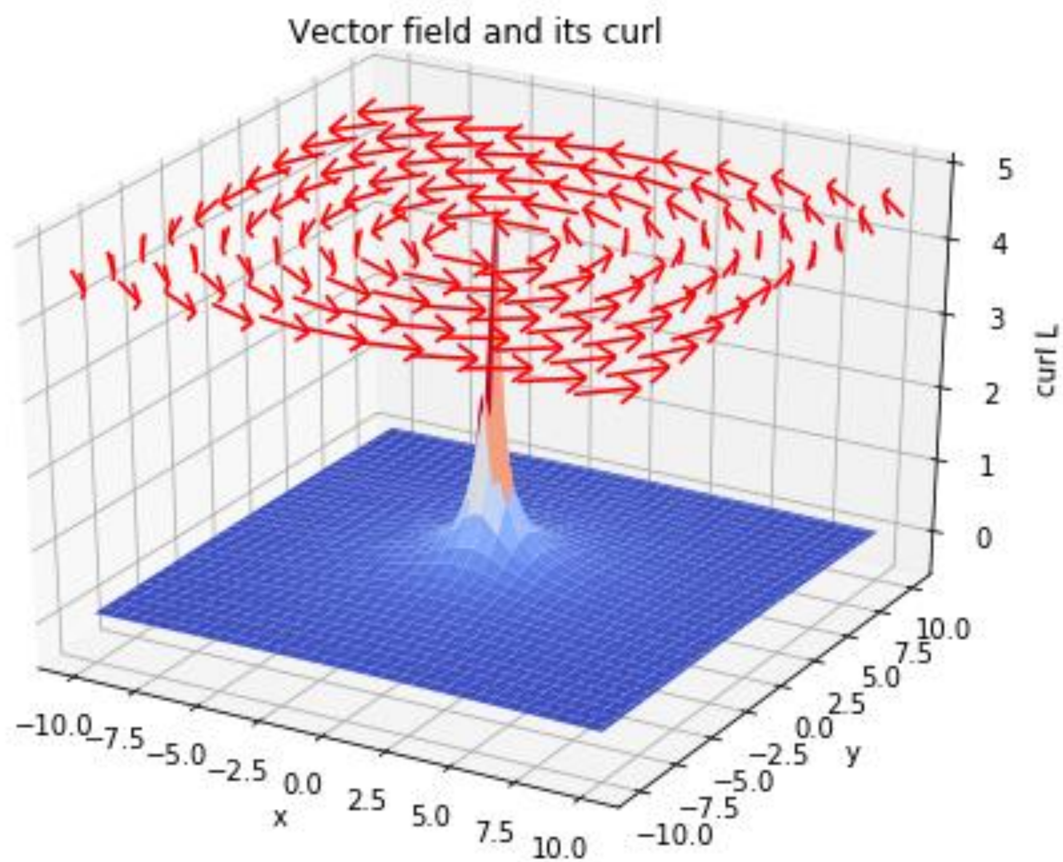
Curl of vector field

- Now field with opposite curl:

- $$\vec{L}(x,y,z) = \begin{pmatrix} \dfrac{-y}{\sqrt{x^2+y^2}} \\ \dfrac{x}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix}$$

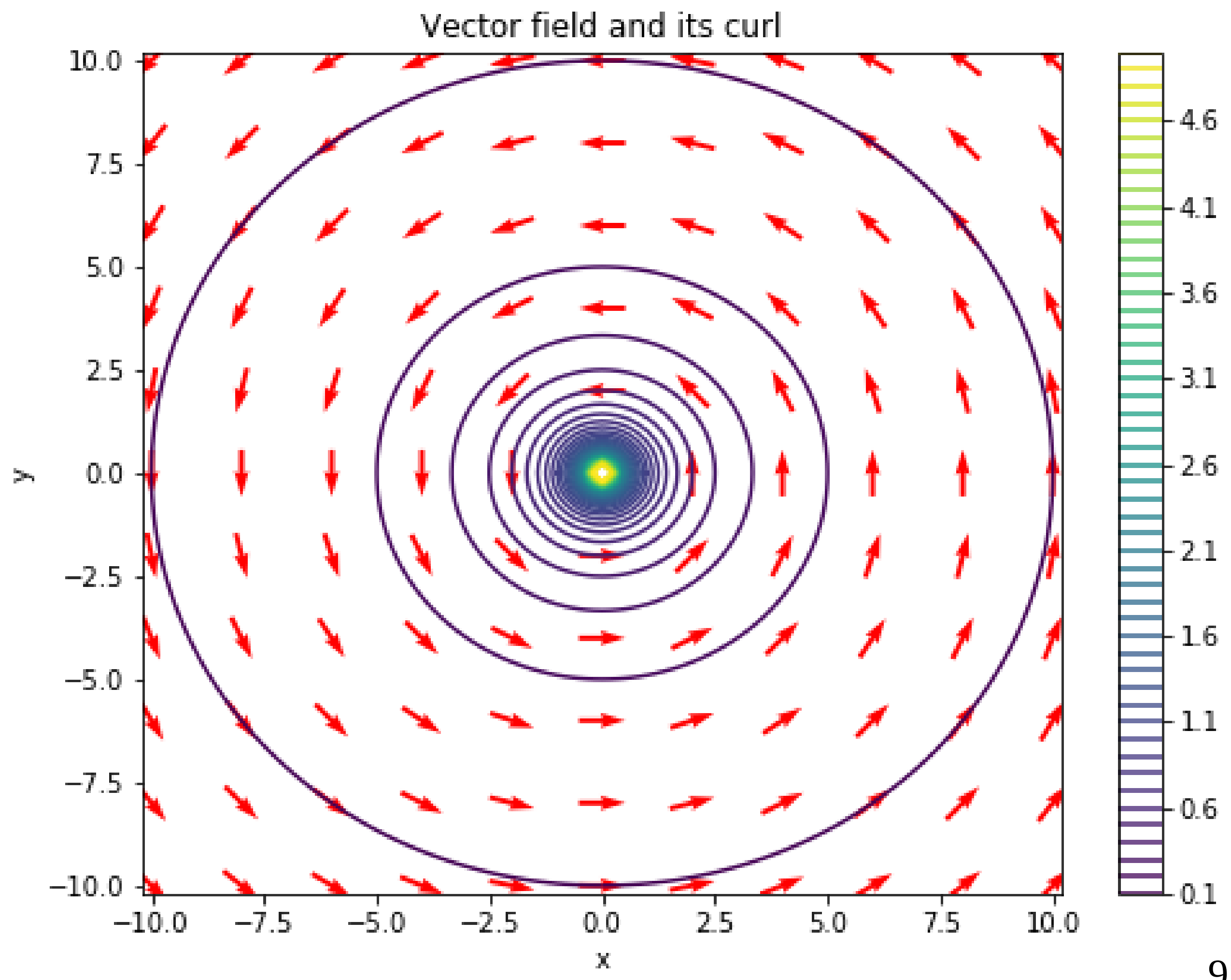
- $$\nabla \times \vec{L}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{\sqrt{x^2+y^2}} \end{pmatrix}$$

- Plotting these quantities:



Curl of vector field

- And again as contour plot:
- $\vec{L}(x,y,z) = \begin{pmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix}$
- $\nabla \times \vec{L}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{x^2+y^2}} \end{pmatrix}$



Curl of electric field

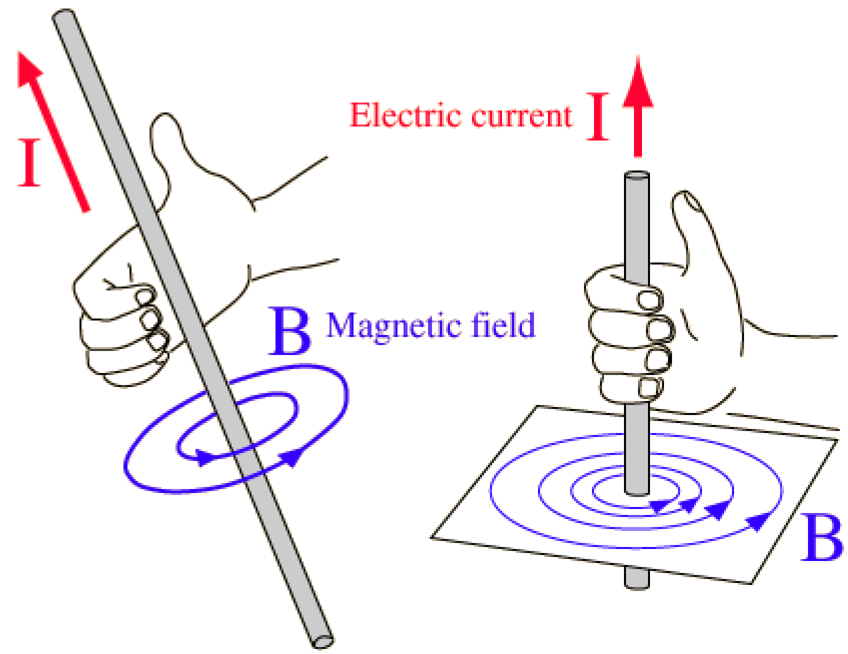
- One of Maxwell's equations (Faraday's Law) involves the curl of the electric field:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

- This implies that changing a magnetic field will cause an electric field to "swirl" around it.
- A further one of Maxwell's equations (Ampere's Law with Maxwell's correction) involves the curl of the magnetic field:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

- Here, $\vec{J}$ is the current density.
- A magnetic field can therefore be induced by an electric current...



- ...or by a changing electric field.
- Changing E fields causes B fields and vice versa, so get waves!

Vector and vector calculus identities

- Some useful vector identities:

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

$$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$$

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

$$(\vec{A}+\vec{B})\cdot\vec{C}=\vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C}$$

$$(\vec{A}+\vec{B})\times\vec{C}=\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D})$$

$$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})=(\vec{A}\cdot(\vec{B}\times\vec{D}))\vec{C}-(\vec{A}\cdot(\vec{B}\times\vec{C}))\vec{D}$$

- Identities for vector calculus:

$$\nabla(\psi+\phi)=\nabla\psi+\nabla\phi$$

$$\nabla(\psi\phi)=\phi\nabla\psi+\psi\nabla\phi$$

$$\nabla(\vec{A}\cdot\vec{B})=(\vec{A}\cdot\nabla)\vec{B}+(\vec{B}\cdot\nabla)\vec{A}+\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$$

$$\nabla\cdot(\vec{A}+\vec{B})=\nabla\cdot\vec{A}+\nabla\cdot\vec{B}$$

$$\nabla\cdot(\phi\vec{A})=\phi\nabla\cdot\vec{A}+A\cdot\nabla\phi$$

$$\nabla\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla\times(\vec{A}+\vec{B})=\nabla\times\vec{A}+\nabla\times\vec{B}$$

$$\nabla\times(\phi\vec{A})=\phi\nabla\times\vec{A}-\vec{A}\times\nabla\phi$$

$$\nabla\times(\vec{A}\times\vec{B})=\vec{A}(\nabla\cdot\vec{B})-\vec{B}(\nabla\cdot\vec{A})+(\vec{B}\cdot\nabla)\vec{A}-(\vec{A}\cdot\nabla)\vec{B}$$

$$\nabla\times(\nabla\phi)=0,\ \ \nabla\cdot(\nabla\times\vec{A})=0$$

Example of proof of vector calculus identity

- Show that $\nabla\times(\nabla\phi)=0$.

$$\nabla\times(\nabla\phi)=\nabla\times\begin{pmatrix}\frac{\partial}{\partial x}\phi & \frac{\partial}{\partial y}\phi & \frac{\partial}{\partial z}\phi\end{pmatrix}$$

$$=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial x}\phi & \frac{\partial}{\partial y}\phi & \frac{\partial}{\partial z}\phi\end{vmatrix}$$

$$=\begin{pmatrix}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\phi-\frac{\partial}{\partial z}\frac{\partial}{\partial y}\phi & \frac{\partial}{\partial z}\frac{\partial}{\partial x}\phi-\frac{\partial}{\partial x}\frac{\partial}{\partial z}\phi & \frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi-\frac{\partial}{\partial y}\frac{\partial}{\partial x}\phi\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\end{pmatrix}.$$