Phys108 – Mathematics for Physicists II

- Lecturer:
  - Prof. Tim Greenshaw.
  - Oliver Lodge Lab, Room 333.
  - Office hours, Fri. 11:30…13:30.
  - Email green@liv.ac.uk
- Lectures:
  - Monday 14:00, HSLT.
  - Wednesday 13:00, HSLT.
  - Thursday 09:00, HSLT.
- Problems Classes:
  - Friday 9:00...11:00.
  - Central Teaching Labs, GFlex.
- Outline syllabus:
  - Matrices.
  - Vector calculus.
  - Differential equations.
  - Fourier series.
  - Fourier integrals.
- Recommended textbook:
  - "Calculus, a Complete Course", Adams and Essex, (Pub. Pearson).
- Assessment:
  - Exam end of S2: 70%.
  - Problems Classes: 20%.
  - Homework: 10%.

Lecture 3 – Matrices

- In this lecture we will:
  - Have a first look at how matrices transform vectors.
  - Introduce eigenvalues and eigenvectors.

- Some comprehension questions for this lecture.
- Find the eigenvalues and eigenvectors of the following matrices:
- $\begin{pmatrix} 2 & 0 \\ -3 & 5 \end{pmatrix}$
- $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Matrices transform vectors

- The following vector defines a position in the (x, y) plane:

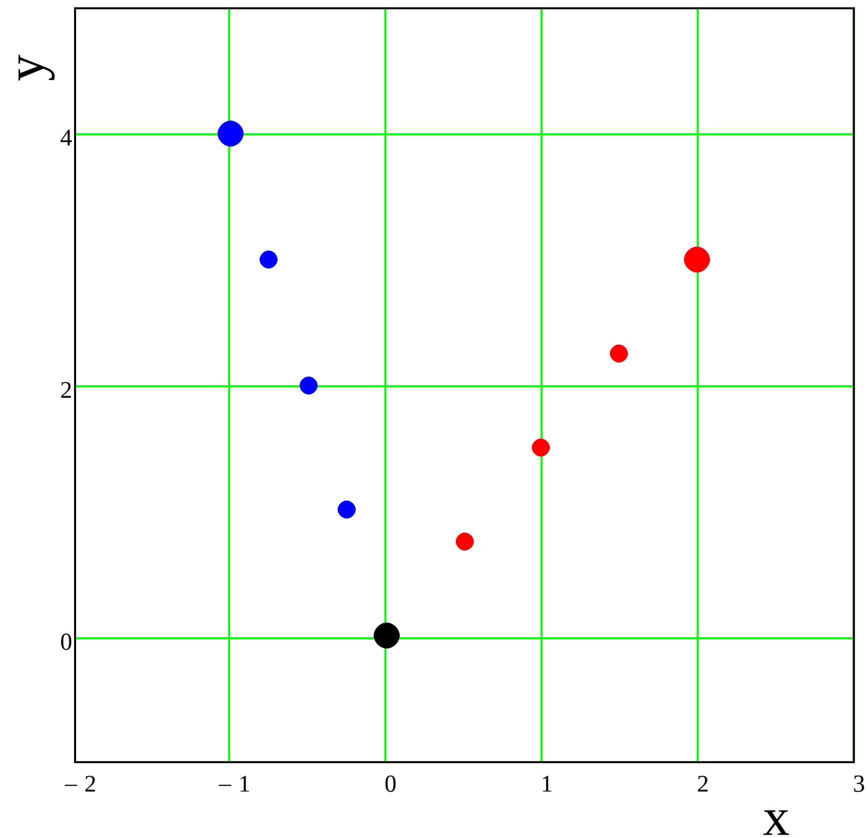
$$\vec{r}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

- If we multiply this vector by a matrix, the position it defines can change:

$$\vec{r}_2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

- In this case, the vector has been stretched and rotated.

- See this in the plot below, in which $\vec{r}_1$ is red and $\vec{r}_2$ blue.



Eigenvalues and eigenvectors

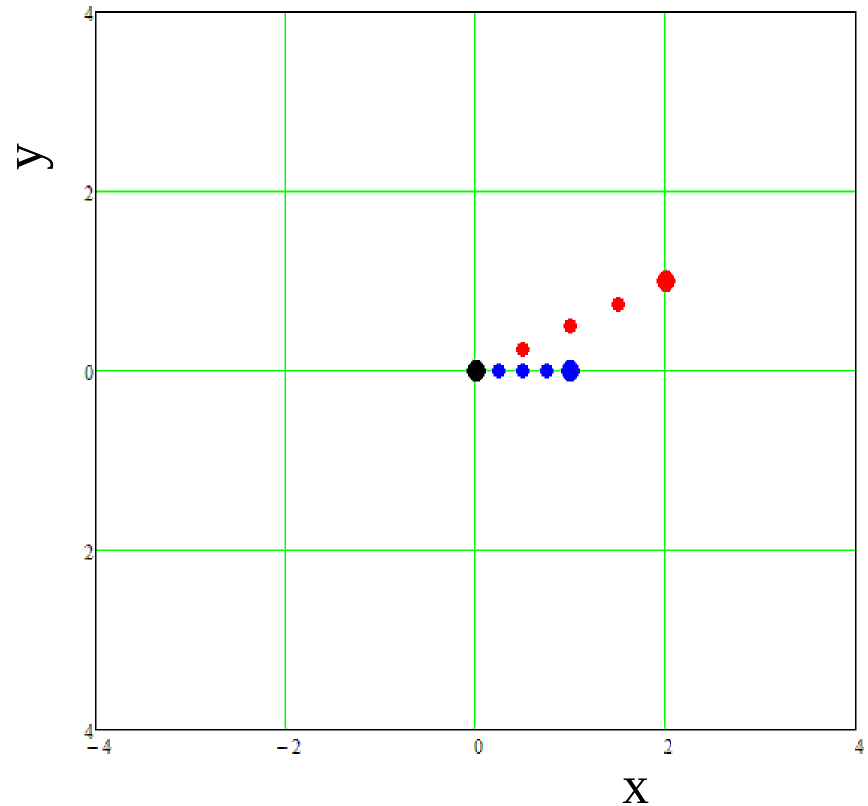
- Matrices can transform/rotate vectors.
- Interesting in quantum mechanics are vectors whose direction is <u>not</u> changed when they are multiplied by a particular matrix (or "operator").
- Look at an example matrix...

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- ...and vector:

$$\vec{r}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

- What happens to $\vec{r}_2$ as θ is changed, i.e. as the vector $\vec{r}_1$ changes direction?
- Look at $\vec{r}_1$ and $\vec{r}_2 = \mathbf{M}\,\vec{r}_1$ as we increase θ from 0 to 9π.



Eigenvalues and eigenvectors

- We see there are some values of θ for which $\vec{r}_1$ and $\vec{r}_2$ are pointing in the same direction (though they may have different lengths).
- These values of $\vec{r}_1$ and $\vec{r}_2$ are called eigenvectors.
- (The German word “eigen” means distinctive or singular.)
- When $\vec{r}_1$ and $\vec{r}_2$ are in the same direction, they must satisfy the equation $\vec{r}_2 = \lambda\,\vec{r}_1$ or $\mathbf{M}\,\vec{r}_1 = \lambda\,\vec{r}_1$.
- The constants λ are the eigenvalues associated with the eigenvectors.
- The eigenvector (or eigenvalue) equation is therefore: $\mathbf{M}\,\vec{x} = \lambda\,\vec{x}$.
- We can rewrite this:
  $\mathbf{M}\,\vec{x} - \lambda\,\vec{x} = \vec{0}$.
- Tempting to then write $(\mathbf{M} - \lambda)\,\vec{x} = \vec{0}$...
- ...but **M** – λ is not defined!
- More correctly:
  $(\mathbf{M} - \lambda\,\mathbf{1})\,\vec{x} = \vec{0}$.
- This is commonly abbreviated to $(\mathbf{M} - \lambda)\,\vec{x} = 0$, on the understanding that there is a suppressed **1** in there...
- ...and that the 0 is in fact the vector:

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Eigenvalues and eigenvectors

- Can we solve $(\mathbf{M} - \lambda\mathbf{1})\vec{x} = \vec{0}$?
- This is a weird equation!
- Write as $\mathbf{A}\vec{x} = \vec{0}$.
- Try and solve by multiplying both sides by $\mathbf{A}^{-1}$.
- Gives $\vec{x} = \mathbf{A}^{-1}\vec{0}$.
- Either:
  - The only eigenvector is $\vec{0}$.
- Or:
  - $\mathbf{A}^{-1}$ doesn't exist.
- We have seen an example with a non-zero eigenvector, so the first alternative is not true...
- How can we have a matrix for which the inverse is not defined?
- If the determinant is zero!
- Hence, we must solve the equation

$$\det(\mathbf{A}) = \det(\mathbf{M} - \lambda\mathbf{1}) = 0.$$

- Look at the example in the video.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

- Then:

$$\mathbf{M} - \lambda\mathbf{1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}.$$

Eigenvalues and eigenvectors

- Hence must solve:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda) - 1 = 0$$

$$\text{or } \lambda^2 - 4\lambda + 3 = 0$$

$$\text{so } (\lambda - 1)(\lambda - 3) = 0$$

$$\text{so } \lambda = 1 \text{ or } 3.$$

- We now have the eigenvalues, how do we find the eigenvectors?
- Two methods possible.
- First: start from $(\mathbf{M} - \lambda\,\mathbf{1})\,\vec{x} = \vec{0}$.
- E.g. for λ = 1,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Hence:

$$x + y = 0$$

$$x + y = 0$$

$$\text{so } y = -x.$$

- Any vector of the form $k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ will do.

E.g. pick "simplest": $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- Second: start from $\mathbf{M}\,\vec{x} = \lambda\,\vec{x}$.
- E.g. for λ = 3,

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues and eigenvectors

- Hence:

$$2x + y = 3x$$
$$x + 2y = 3y$$
$$\text{so } y = x.$$

- Any vector of the form $k\begin{pmatrix}1\\1\end{pmatrix}$ will do.

Pick: $\vec{x}_2 = \begin{pmatrix}1\\1\end{pmatrix}.$

- Check by substitution, e.g. for $\lambda = 1$:

$$\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = 1\begin{pmatrix}1\\-1\end{pmatrix}$$

- Some useful results:
- Determinant of matrix is equal to product of eigenvalues.
  - $|\mathbf{M}| = 3.$
  - $\prod_{i=1}^{2}\lambda_i = 1\times 3 = 3.$
- Sum of diagonal elements of **M** – the *trace* of **M**, $\mathrm{Tr}(\mathbf{M})$ – is equal to sum of eigenvalues.
  - $\mathrm{Tr}(\mathbf{M}) = \sum_{i=1}^{2} M_{ii} = 2 + 2 = 4.$
  - $\sum_{i=1}^{2}\lambda_i = 1 + 3 = 4.$

Eigenvalues and eigenvectors

- Eigenvalues of diagonal matrix are elements on the diagonal.

  E.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

- Must solve:

  $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

  i.e. $(1-\lambda)(2-\lambda)(3-\lambda) = 0$

  so $\lambda = 1,\ 2$ or 3.

- What are the eigenvectors in this case?

- E.g. look at $\lambda = 2$.

  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

  i.e. $x = 2x,\ 2y = 2y$ and $3z = 2z$.

- Looks strange, no constraint for y!

- Solution, x = 0, z = 0 and y allowed to have any value, e.g. pick y = 7.

- Check: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}$

  $$\begin{pmatrix} 0 \\ 14 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ 0 \end{pmatrix}.$$