Phys108 – Mathematics for Physicists II

- Lecturer:
  - Prof. Tim Greenshaw.
  - Oliver Lodge Lab, Room 333.
  - Office hours, Fri. 11:30…13:30.
  - Email green@liv.ac.uk
- Lectures:
  - Monday 14:00, HSLT.
  - Wednesday 13:00, HSLT.
  - Thursday 09:00, HSLT.
- Problems Classes:
  - Friday 9:00...11:00.
  - Central Teaching Labs, GFlex.
- Outline syllabus:
  - Matrices.
  - Vector calculus.
  - Differential equations.
  - Fourier series.
  - Fourier integrals.
- Recommended textbook:
  - "Calculus, a Complete Course", Adams and Essex, (Pub. Pearson).
- Assessment:
  - Exam end of S2: 70%.
  - Problems Classes: 20%.
  - Homework: 10%.

Lecture 2 – Matrices

- In this lecture we will:
  - Introduce the transpose.
  - Look at determinants.
  - Define minors and cofactors.
  - Define the adjugate and inverse of a matrix.
  - Use matrices to solve simultaneous equations.
  - Introduce Cramer's Rule.
- Some comprehension questions for this lecture.
- Find the adjugate of:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

- Calculate the determinant of **A** and hence find $\mathbf{A}^{-1}$.
- Use the above to solve the simultaneous equations:

$$\begin{aligned} x + 3y &= 1 \\ 2x - 2y - z &= 3 \\ x - y + 2z &= 0 \end{aligned}$$

The transpose

- We may need to switch the rows and columns of vectors and matrices, i.e. form the transpose.
- $A_{i\,j}{}^T = A_{j\,i}$.
- Example:

$$\begin{pmatrix} 12 & 21 \\ 13 & 17 \\ 18 & 19 \end{pmatrix}^T = \begin{pmatrix} 12 & 13 & 18 \\ 21 & 17 & 19 \end{pmatrix}$$

- Can use to give dot product of vectors.
- E.g. for two row vectors $\vec{r}_1$ and $\vec{r}_2$,

$$\vec{r}_1 \cdot \vec{r}_2 = \vec{r}_1\ \vec{r}_2{}^T.$$

- Example:
- $\vec{r}_1 = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$, $\vec{r}_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$

$$\vec{r}_1 \cdot \vec{r}_2 = 1\times 1 + (-2\times 0) + 3\times 1 = 4.$$

- $\vec{r}_2{}^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4$$

- Note, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$!

Matrices and determinants

- Work now only with square matrices.
- The determinant of a 2 × 2 matrix is:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
$$= ad - bc.$$

- Example:

$$\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 1 \times (-2) - 2 \times 0$$
$$= -2$$

- We can build up the determinant of a larger (square!) matrix iteratively.
- The determinant of a 3 × 3 matrix is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrices and determinants

- Write down an expression for the determinant of the following 4 × 4 matrix in terms of 3 × 3 determinants.

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} =$$

Minors and cofactors

- The minor $M_{i\,j}$ of an element $A_{i\,j}$ of an n × n matrix **A** is the n – 1 × n – 1 determinant obtained when the i^th row and the j^th column are removed from **A**.
- Find minor of (1, 2) element of:

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

$$\text{Remove row 1, col 2} \begin{pmatrix} \times & \times & \times \\ d & \times & f \\ g & \times & i \end{pmatrix}$$

$$\text{Hence } M_{12} = \begin{vmatrix} d & f \\ g & i \end{vmatrix}.$$

- The cofactor $C_{i\,j}$ of $A_{i\,j}$ is given by:
  $C_{i\,j} = -1^{i+j} M_{i\,j}$
- Cofactors alternate sign across rows and down columns.
- For our 3 × 3 matrix, we have

$$C_{12} = -M_{12} = -\begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

- Putting these definitions together we see that the determinant is given by:

$$|\mathbf{A}| = A_{11}M_{11} - A_{12}M_{12} + A_{13}M_{13}$$
$$= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}$$

- Show that

$$A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}$$
$$= A_{11}C_{11} + A_{21}C_{21} + A_{31}C_{31} = |\mathbf{A}|.$$

Adjugate and inverse of a matrix

- The adjugate of a matrix is the transpose of the matrix of cofactors, e.g. adj(**A**) = $\mathbf{C}^T$, where **C** is the matrix of cofactors of **A**.
- Useful as it allows us to determine the inverse...
- The inverse of a matrix is the adjugate matrix divided by the determinant.
- If $\Delta = |\mathbf{A}|$, the components of $\mathbf{A}^{-1}$ are given by:

$$A^{-1}{}_{ij} = \frac{1}{\Delta} C_{ji}.$$

- Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\det(\mathbf{A}) = -2,$$

$$\operatorname{cof}(\mathbf{A}) = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\operatorname{adj}(\mathbf{A}) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Identity matrix and inverse of a matrix

- The product $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{1}$, where: $\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}.$

- Check for $\mathbf{A}$ as defined above: $$\mathbf{AA}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Note, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$!

Identity matrix and inverse of a matrix

- Exercises:
- Show that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{1}$.
- Determine the inverse of the matrices:

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}.$$

- Prove that **B** is the inverse of **A**, where:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ -2 & 2 & 1 \end{pmatrix}, \ \mathbf{B} = \frac{1}{4}\begin{pmatrix} 1 & 3 & -2 \\ 2 & 2 & 0 \\ -2 & 2 & 0 \end{pmatrix}$$

- What is the inverse of **B**?

Solving simultaneous equations using matrices

- Matrices are extremely useful!
- One application:
  solving simultaneous equations.
- Consider:

$$\begin{aligned} x + y - z &= 1 \\ -x + y + z &= 3 \\ -2x + y + 3z &= -2 \end{aligned}$$

- Can write as matrix equation $\mathbf{A}\vec{x} = \vec{c}$, where:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

- Multiplying the matrix equation from the left by $\mathbf{A}^{-1}$ gives:

$$\mathbf{A}^{-1}\mathbf{A}\vec{x} = \mathbf{A}^{-1}\vec{c}$$

$$\Rightarrow \vec{x} = \mathbf{A}^{-1}\vec{c}.$$

- From this can read off the values of x, y and z.
- Here,

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix}, \quad A^{-1}\vec{c} = \begin{pmatrix} -7 \\ 2 \\ -6 \end{pmatrix}$$

Hence:

$$\begin{aligned} x &= -7 \\ y &= 2 \\ z &= -6 \end{aligned}$$

Solving simultaneous equations using Cramer's Rule

- Consider same set of equations:

$$x + y - z = 1$$
$$-x + y + z = 3$$
$$-2x + y + 3z = -2$$

- Provided the determinant Δ of the coefficient matrix **A** is not zero, the solution is given by:

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}.$$

- Here,

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ -2 & 1 & 3 \end{vmatrix}$$

- and,

$$\Delta_2 = \begin{vmatrix} 1 & 1 & -1 \\ -1 & 3 & 1 \\ -2 & -2 & 3 \end{vmatrix},\ \Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

- Hence, e.g.

- $$x = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ -2 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{vmatrix}} = -7$$

Examples

- Write down the transpose of the matrix:

$$\begin{pmatrix} 11 & 21 & 31 \\ 21 & 22 & 32 \\ 31 & 23 & 33 \\ 41 & 24 & 34 \end{pmatrix}$$

- Prove that, for any 2 × 2 matrices **A** and **B**, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

- Prove that:

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$$

for the 3 × 3 diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$