Phys108 – Mathematics for Physicists II

- Lecturer:
  - Prof. Tim Greenshaw.
  - Oliver Lodge Lab, Room 333.
  - Office hours, Fri. 11:30…13:30.
  - Email green@liv.ac.uk
- Lectures:
  - Monday 14:00, HSLT.
  - Wednesday 13:00, HSLT.
  - Thursday 09:00, HSLT.
- Problems Classes:
  - Friday 9:00...11:00.
  - Central Teaching Labs, GFlex.
- Outline syllabus:
  - Matrices.
  - Vector calculus.
  - Differential equations.
  - Fourier series.
  - Fourier integrals.
- Recommended textbook:
  - "Calculus, a Complete Course", Adams and Essex, (Pub. Pearson).
- Assessment:
  - Exam end of S2: 70%.
  - Problems Classes: 20%.
  - Homework: 10%.

Lecture 1 – Matrices

- In this lecture we will:
  - Motivate the introduction of matrices.
  - Look at matrix addition.
  - Look at multiplication of matrices by a scalar.
  - Look at multiplication of two matrices.

- Some comprehension questions:
- What is the value of the component in row 2 and column 3 of the following matrix?

$$\begin{pmatrix} -1 & 1 & 2 & 0 \\ 3 & 0 & -3 & -5 \\ -2 & -4 & 0 & 6 \end{pmatrix}$$

- What is the order of this matrix?
- Calculate the following:

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix} =$$

Motivating matrices – addition

- Tables of numbers are often useful.
- E.g. number of apples and bananas Alan, Bob and Catherine eat on Monday...

Fruit Monday	Apples	Bananas
Alan	1	4
Bob	0	5
Catherine	3	2

- ...and on Tuesday.

Fruit Tuesday	Apples	Bananas
Alan	3	2
Bob	5	0
Catherine	3	2

- How much have they eaten in total?

Mon + Tues	Apples	Bananas
Alan	4	6
Bob	5	5
Catherine	6	4

- Have “table addition rule”:

$$\begin{pmatrix} 1 & 4 \\ 0 & 5 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 5 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 5 & 5 \\ 6 & 4 \end{pmatrix}.$$

- Only works if tables have same number of rows and columns!

Motivating matrices – multiplication

- Another way of using tables:
- Number of apples and bananas Alan, Bob and Catherine eat in a week:

Fruit in week	Apples	Bananas
Alan	12	21
Bob	13	17
Catherine	18	19

- Cost of apples and bananas:

Fruit	Cost (£)
Apples	0.50
Bananas	0.80

- How much does each person spend on fruit in a week?
  - Alan: $12 \times 0.5 + 21 \times 0.8 = 22.8$
  - Bob: $13 \times 0.5 + 17 \times 0.8 = 20.1$
  - Cath: $18 \times 0.5 + 19 \times 0.8 = 24.2$
- See we need "table multiplication rule":

$$\begin{pmatrix} 12 & 21 \\ 13 & 17 \\ 18 & 19 \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 22.8 \\ 20.1 \\ 24.2 \end{pmatrix}.$$

- Position in table is crucial, determines what numbers refer to.
- Number of columns in first table same as number of rows in second.

Motivating matrices – more multiplication

- More complicated problem: saving money by buying unripe fruit.
- Number of apples and bananas Alan, Bob and Catherine eat in a week:

Fruit in week	Apples	Bananas
Alan	12	21
Bob	13	17
Catherine	18	19

- Cost of ripe and unripe fruit:

Fruit	Cost ripe	Cost unripe
Apples	0.50	0.30
Bananas	0.80	0.40

- What would each person have to spend a week if they bought ripe or unripe fruit?
- Use table multiplication rule twice:

$$\begin{pmatrix} 12 & 21 \\ 13 & 17 \\ 18 & 19 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.3 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 22.8 & 12.0 \\ 20.1 & 10.7 \\ 24.2 & 13.0 \end{pmatrix}.$$

- Again, only works if number of columns in first table is same as number of rows in second!
- How would we determine the cost per person if they bought either ripe or unripe fruit for four weeks?

Examples

- Try the following:
- $\begin{pmatrix} 1 & 2 & 4 \\ 5 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 2 \\ 2 & -3 & -2 \end{pmatrix} =$
- $\frac{1}{2}\begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix} =$
- $\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & -1 \end{pmatrix} =$
- $\begin{pmatrix} 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} =$

Introducing matrices – addition

- These tables are of course matrices.
- A matrix with one row is called a row vector...

$$\vec{r} = \begin{pmatrix} a & b & c & d \end{pmatrix}$$

- ...with one column a column vector...

$$\vec{c} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- ...and with m rows and n columns an m × n matrix.

$$\mathbf{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$$

- The dimensions define the order of the matrix (i.e. m × n).
- Matrices are equal if are of same order and all components are same.
- Can add matrices if are of same order.
- Addition performed on corresponding components:

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} + \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} + B_{11} & \cdots & A_{1n} + B_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} + B_{m1} & \cdots & A_{mn} + B_{mn} \end{pmatrix}$$

Introducing matrices – multiplication

- Matrices can be multiplied by a scalar:

$$k\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} = \begin{pmatrix} k\,A_{11} & \cdots & k\,A_{1n} \\ \vdots & \ddots & \vdots \\ k\,A_{m1} & \cdots & k\,A_{mn} \end{pmatrix}$$

- The product, **AB**, of two matrices **A** and **B** exists if the number of columns in **A** is the same as the number of rows in **B**.
- Rule for multiplication of an m × p matrix by a p × n matrix to give a matrix of order m × n:

$$AB_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$$

- E.g. for two 2 × 2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

- Einstein summation convention: sometimes omit "Σ" and assume summation over repeated indices (common in books on General Relativity).

$$AB_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$$

$$\to AB_{ij} = A_{ik} B_{kj}$$

Examples

- Given matrices **A**, **B** and **C** which satisfy **C** = **A** + **B**, which of the following statements is correct?
- $C_{i\,j} = A_{i\,j} + B_{j\,i}$.
- $C_{i\,k} = A_{i\,k} + B_{i\,k}$.
- Matrices **E**, **F** and **G** have order 2 × 2, 2 × 4 and 4 × 2, respectively. Which of the following quantities is defined, **EF**, **EG**, **FG**?
- Express the following matrix as a scalar multiplied by a matrix:

$$\begin{pmatrix} 3 & -6 \\ -9 & 3 \\ -6 & 12 \end{pmatrix}$$

Examples

- Multiply the following matrix and vector:

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Write these simultaneous equations as a matrix multiplying a vector:
- $$\begin{aligned} 2x - y &= 12 \\ -3x + 2y &= 7 \end{aligned}$$
- $$\begin{aligned} 2y - x + 2z &= 12 \\ -3x + 2y - z + 2 &= 7 \\ z - x + 5y &= 0 \\ 2y - z &= -3 \end{aligned}$$

- Is matrix addition commutative, i.e. does **A** + **B** = **B** + **A**?
- Is matrix multiplication commutative?
- Show matrix multiplication and addition are associative (i.e. **A**(**BC**) = (**AB**)**C** etc.) for the matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}$$

- Show also that matrix multiplication is distributive over matrix addition for the three matrices **A**, **B** and **C**, i.e. **A**(**B** + **C**) = **AB** + **AC** and (**A** + **B**)**C** = **AC** + **BC**.