Differential equations

- In this lecture we will:
  - Find out how to solve various types of inhomogeneous second order differential equation.
- Some comprehension questions for this lecture.
  - Find the general solution of the equations:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=x^2+5$$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=5e^{-3x}$$

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Inhomogeneous second order differential equations

- Here, we look at inhomogeneous (or non-homogeneous) second order differential equations, i.e. equations of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y=f(x)$$

- The homogenous differential equation obtained by setting $f(x) = 0$,

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y=0,$$

with the same coefficients as the above, is called the complementary equation.
- Suppose the general solution (containing arbitrary constants) of the complementary equation is $y_c(x)$ and that a particular solution (no arbitrary constants) of the inhomogeneous equation is $y_p(x)$.
- We can then show that $y(x)=y_c(x)+y_p(x)$ is a general solution of the inhomogeneous equation.
- Do this by writing the solution of the complementary equation in the form $y_c(x) = y(x) - y_p(x)$.

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- Then, substituting for $y_c(x)$:

$$a\frac{d^2}{dx^2}(y-y_p)+b\frac{d}{dx}(y-y_p)+c(y-y_p)$$

$$=a\frac{d^2}{dx^2}y_c+b\frac{d}{dx}y_c+cy_c$$

$$\Rightarrow a\frac{d^2}{dx^2}y+b\frac{d}{dx}y+cy$$

$$-\left(a\frac{d^2}{dx^2}y_p+b\frac{d}{dx}y_p+cy_p\right)=0$$

$$\Rightarrow a\frac{d^2}{dx^2}y+b\frac{d}{dx}y+cy$$

$$=a\frac{d^2}{dx^2}y_p+b\frac{d}{dx}y_p+cy_p=f(x)$$

- Hence, if we can find a general solution of the complementary equation, $y_c(x)$, and a particular solution (particular integral) of the inhomogeneous equation, their sum will be a general solution of the inhomogeneous equation.
- We already know how to find solutions of homogeneous equations with constant coefficients.
- How can we find particular solutions of inhomogeneous equations (again restricted to constant coefficients)?
- Educated guesswork...also known as the method of undetermined coefficients.

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Inhomogeneous second order differential equations

- An example:
- Find the general solution of the equation:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=x^2$$

- The complementary equation is...

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

- ...and the associated auxiliary equation:

$$m^2+m-2=0$$

$$\Rightarrow (m-1)(m+2)=0$$

$$m_1=1,\ \ m_2=-2$$

- Hence the general solution of the complementary equation is

$$y_c(x)=C_1e^x+C_2e^{-2x}$$

- Since $f(x) = x^2$ and differentiating this will give both a term in x and a constant, we try the particular solution

$$y_p(x)=Ax^2+Bx+C$$

- The values of A, B and C can be determined by substituting into the inhomogeneous equation.
- We need y_p and its differentials:

$$\frac{dy_p}{dx}=2Ax+B \text{ and } \frac{d^2y_p}{dx^2}=2A.$$

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- Hence:
$$(2A)+(2Ax+B)-2(Ax^2+Bx+C)=x^2$$
$$\Rightarrow -2Ax^2+(2A-2B)x+2A+B-2C=x^2$$
- For this to hold for all x, must have:
$-2A=1$ [coefficents of x^2]
$2A-2B=0$ [coefficents of x]
and $2A+B-2C=0$ [constants].
- Hence:
$A=-1/2$, $2B=2A\Rightarrow B=-1/2$ and
$C=\frac{2A+B}{2}=-\frac{3}{4}$.
- Our particular solution is thus:
$$y_p=-\frac{1}{2}x^2-\frac{1}{2}x-\frac{3}{4}.$$
- And the general solution is:
$$y(x)=y_c(x)+y_p(x)=C_1e^x+C_2e^{-2x}-\frac{1}{2}x^2-\frac{1}{2}x-\frac{3}{4}.$$
- Note, we still have two arbitrary constants, the values of which can be determined using initial conditions.
- This illustrates how inhomogeneous differential equations can be solved if $f(x)$ is a polynomial.
- There is one possible difficulty...

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Inhomogeneous second order differential equations

- Find a particular solution to the differential equation:
$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=5$$
- Try $y_p=C$
$$\Rightarrow \frac{dy_p}{dx}=0$$
- Cannot substitute to work out coefficients.
- Must modify trial function to ensure we get required behaviour.
- In general, multiply y_p by x, e.g. if trial function x^2+2x+1 doesn't work, try $x(x^2+2x+1)$.
- Example here, try $y_p(x)=Cx$.
- Then have:
$$\frac{dy_p}{dx}=C \text{ and } \frac{d^2y_p}{dx^2}=0.$$
- Hence $C=5$ and $y_p(x)=5x$.
- The auxiliary equation is $m(m+1)=0$ so the general solution of the complementary equation is:
$$y_c=C_1e^0+C_2e^{-x}=C_1+C_2e^{-x}$$
- The general solution of the inhomogeneous equation is therefore
$$y(x)=C_1+C_2e^{-x}+5x$$
- C_1 and C_2 can then be determined using the initial conditions.

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- Now consider case that $f(x)$ is of the form $Ae^{\gamma x}$.
- The trial function depends on the roots of the auxiliary equation.
  - If $m_1, m_2\neq\gamma$, try $y_p=Ae^{\gamma x}$.
  - If $m_1=\gamma$, $m_2\neq\gamma$, try $y_p=Axe^{\gamma x}$.
  - If $m_1=m_2=\gamma$, try $y_p=Ax^2e^{\gamma x}$.
- Example:
- Find a particular solution to:
$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=3e^{-x}$$
- The auxiliary equation
$$m^2+2m+1=0$$
$$\Rightarrow (m+1)^2=0 \text{ and } m_1=m_2=-1$$
- Hence we try:
$$y_p=Ax^2e^{-x}$$
$$\frac{dy_p}{dx}=-Ax^2e^{-x}+2Axe^{-x}$$
$$\frac{d^2y_p}{dx^2}=Ax^2e^{-x}-2Axe^{-x}-2Axe^{-x}+2Ae^{-x}=Ax^2e^{-x}-4Axe^{-x}+2Ae^{-x}$$
- Substituting gives:
$$\left(Ax^2e^{-x}-4Axe^{-x}+2Ae^{-x}\right)+2\left(-Ax^2e^{-x}+2Axe^{-x}\right)+Ax^2e^{-x}=3e^{-x}$$
- Similar to previous case, compare coefficients, in this case of e^{-x}, xe^{-x} and x^2e^{-x}, to determine A.

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Inhomogeneous second order differential equations

- Hence $2Ae^{-x}=3e^{-x}$ and $A=3/2$.
- The solution of the complementary equation is $y_c=C_1e^{-x}+C_2xe^{-x}$ giving a general solution of the inhomogeneous equation
$$y=C_1e^{-x}+C_2xe^{-x}+\frac{3}{2}x^2e^{-x}.$$

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