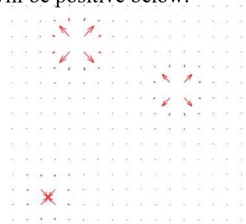
Vector calculus

- In this lecture we will:
  - Define the divergence of a vector field.
  - Look at some examples to try and gain some insight into what the divergence represents.
  - Discuss the divergence of the electric and magnetic fields.
- Some comprehension questions for this lecture.
  - Indicate where the divergence will be positive below.



  - Calculate the divergence of the field $\vec{F}(x,y,z) = \begin{pmatrix} x^2 & 2xy & xyz \end{pmatrix}$.

1

1

Divergence of a vector field

- The divergence of a vector field is defined by the expression:

$$\nabla \cdot \vec{F}(x,y,z) = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z$$

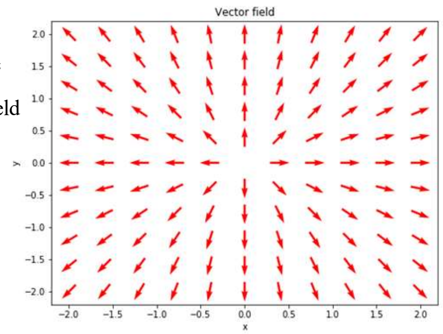
- Note, the divergence of a vector field is a scalar field.
- Can think of the divergence as dot product of vector with the operator

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}.$$

- Look at an example (in 2D so can visualise more easily):

$$\vec{F}(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix}$$

- Plot vector field $\vec{F}(x,y)$:



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2

Divergence of a vector field

- Calculate divergence, x component...

$$\frac{\partial}{\partial x}F_x = \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{\sqrt{x^2+y^2} - x \times \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x}{x^2+y^2}$$

$$= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

- ...and y component

$$\frac{\partial}{\partial y}F_y = \frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

- Putting these together:

$$\nabla \cdot \vec{F} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

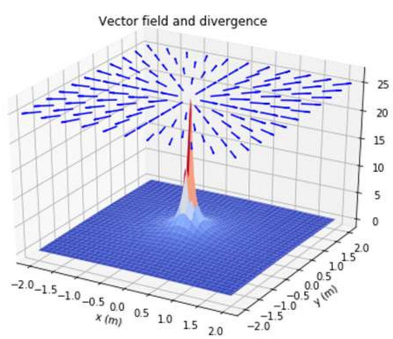
$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

3

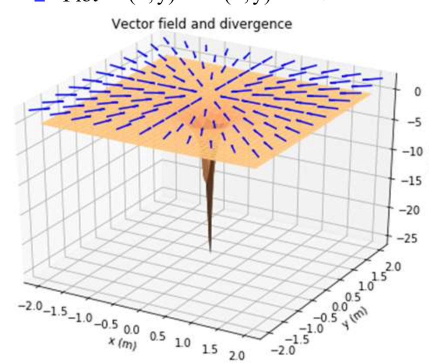
3

Divergence of a vector field

- Plot $\vec{F}$ and $\nabla \cdot \vec{F}$:



- Plot $\vec{G}(x,y) = -\vec{F}(x,y)$ and $\nabla \cdot \vec{G}$:



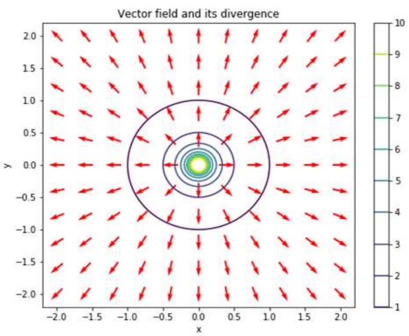
- Divergence of field reveals “sources” (left) and “sinks” (right) of field.

4

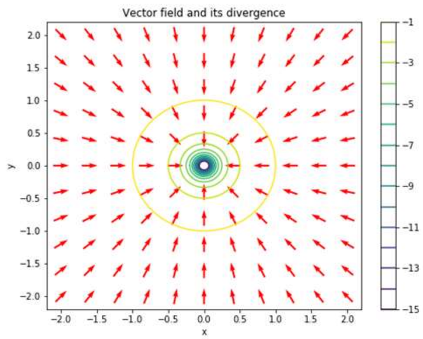
4

Divergence of a vector field

- Plot $\vec{F}$ and $\nabla \cdot \vec{F}$ using contours:



- Plot $\vec{G}(x,y) = -\vec{F}(x,y)$ and $\nabla \cdot \vec{G}$:



Divergence of a field

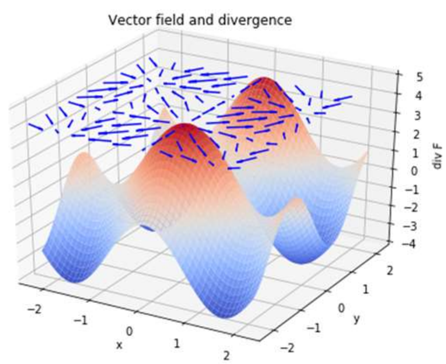
- A field is defined by the equation: $\vec{F}(x,y,z) = \begin{pmatrix} 3\sin x & 2\cos x & -z^2 \end{pmatrix}$.
- Determine the divergence of the field.
- What is the value of $\nabla \cdot \vec{F}(\frac{\pi}{2}, 0, -1)$?

Divergence of a vector field

- Look for sources and sinks in another vector field:
- $\vec{F}(x,y) = \begin{pmatrix} \sin 2x \\ \cos 2y \end{pmatrix}$
- Divergence of this field is:

$$\nabla \cdot \vec{F}(x,y) = \frac{\partial}{\partial x}\sin 2x + \frac{\partial}{\partial y}\cos 2y$$
$$= 2\cos 2x - 2\sin 2y$$

- Plot $\vec{F}$ and $\nabla \cdot \vec{F}$:

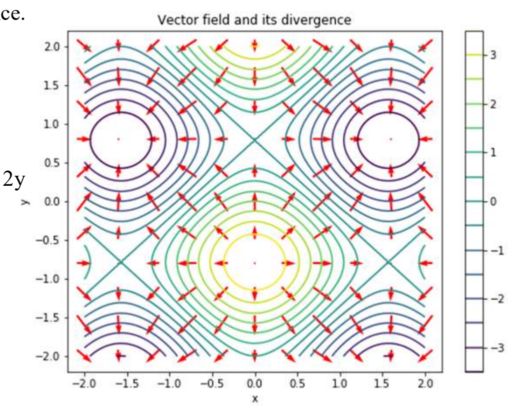


Divergence of a vector field

- Same as previous slide using contours to represent divergence.
- $\vec{F}(x,y) = \begin{pmatrix} \sin 2x \\ \cos 2y \end{pmatrix}$
- Divergence is:

$$\nabla \cdot \vec{F}(x,y) = \frac{\partial}{\partial x}\sin 2x + \frac{\partial}{\partial y}\cos 2y$$
$$= 2\cos 2x - 2\sin 2y$$

- Plot $\vec{F}$ and $\nabla \cdot \vec{F}$:

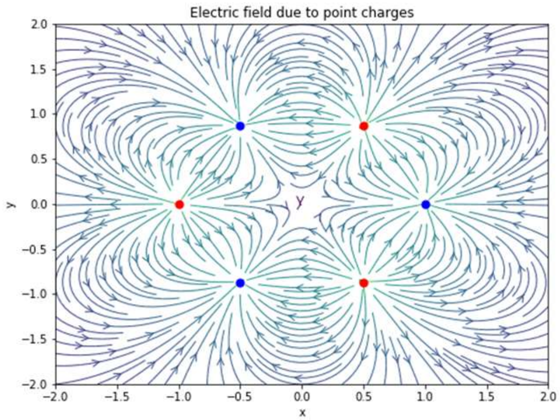


Divergence of electric field

- One of Maxwell's equations (Gauss' Law) involves divergence of E field:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

- The quantity ρ is the density of electric charge.
- This equation is telling as that electric charge is the source of the electric field.



9

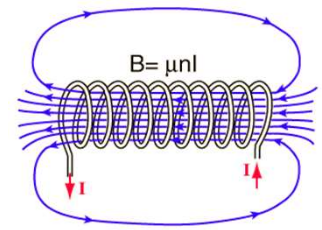
9

Divergence of electric field

- Another of Maxwell's equations (Gauss' Law for magnetism) involves the divergence of the magnetic field:

$$\nabla \cdot \vec{B} = 0$$

- What does this equation tell us about the sources and sinks of the magnetic field?
- And about magnetic monopoles?
- Example of how magnetic field can be generated:



- How is magnetism caused in materials, e.g. "magnets"?
- In the earth?
- Reversal of Earth's magnetic field.

10

10